UNIVERSIDADE FEDERAL DE MINAS GERAIS
Instituto de Ciências Exatas
Programa de Pós-Graduação em Ciência da Computação

Henrique Soares Assumpção e Silva

Semidefinite Optimization on Laplacians of Highly Regular Graphs

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Para Valeria, Julia, Fernando e Darcy

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“[...] As the seasons passed and his missions continued, Marco mastered the Tartar language and the national idioms and tribal dialects. Now his accounts were the most precise and detailed that the Great Khan could wish and there was no question or curiosity which they did not satisfy. And yet each piece of information about a place recalled to the emperor’s mind that first gesture or object with which Marco had designated the place. The new fact received a meaning from that emblem and also added to the emblem a new meaning. Perhaps, Kublai thought, the empire is nothing but a zodiac of the mind’s phantasms.

“On the day when I know all the emblems,” he asked Marco, “shall I be able to posses my empire, at last?”

And the Venetian answered: “Sire, do not believe it. On that day you will be an emblem among emblems.””

(Italo Calvino, Invisible Cities)

Resumo

Esta tese estuda dois temas interconectados em otimização e combinatória: a rigidez espectral de grafos com pesos nas arestas com respeito à matriz Laplaciana, e a computação de limitantes via programação semidefinida para problemas de otimização combinatória em grafos altamente regulares.

Na primeira parte, introduzimos e caracterizamos a noção de *rigidez conformal total*, generalizando os conceitos de Steinerberger e Thomas [70] de autovalores individuais para somas de autovalores do Laplaciano. Mostramos que a rigidez conformal total é equivalente à *rigidez de arestas*, a condição de que todo mergulho espectral canônico de um grafo em um autoespaço do Laplaciano seja isométrico nas arestas. Esta equivalência é estabelecida por meio da análise de programas semidefinidos que modelam somas de autovalores do Laplaciano, e é ainda conectada à coespectralidade de arestas do Laplaciano, à estrutura de caminhadas em grafos linha com sinais, e à teoria de dualidade de gauges. Mostramos também uma caracterização completa: um grafo é aresta-rígido se e somente se é 1-walk-regular ou 1-walk-biregular. Como consequências combinatórias da rigidez conformal total, derivamos que o grafo sem pesos maximiza o número de árvores geradoras e minimiza o índice de Kirchhoff sobre todas as ponderações válidas das arestas. Na segunda parte, estudamos relaxações semidefinidas para os problemas de FCC e MAX 2-SAT em grafos com alta regularidade estrutural. Combinando a teoria de esquemas de associação e dualidade de gauges, estabelecemos casos de igualdade para uma desigualdade que relaciona a formulação SDP do MAXCUT com seu dual, e fornecemos expressões analíticas explícitas para soluções ótimas dos programas semidefinidos em termos das matrizes de autovalores do esquema de associação subjacente.

As duas partes desta tese estão unidas por um fio condutor algébrico comum: a teoria de *dualidade de gauges*. Na primeira parte, ela fornece uma caracterização intrínseca da rigidez conformal total. Na segunda parte, essa mesma perspectiva ressurge no estudo do formulação SDP do MAXCUT e de seu dual, onde a desigualdade fundamental $\eta(G)\eta^{\circ}(G) \geq |E|$ se torna uma igualdade precisamente para grafos com regularidade suficiente, incluindo todos os grafos totalmente conformalmente rígidos. A dualidade de gauges serve, portanto, tanto como uma técnica de demonstração quanto como uma ponte conceitual, revelando que a rigidez espectral de um grafo sob a reponderação do Laplaciano está intimamente conectada a certos limitantes de programas semidefinidos.

Palavras-chave: Teoria dos Grafos, Combinatória Algébrica, Otimização Semidefinida.

Abstract

This thesis studies two interconnected themes related to semidefinite optimization and spectral graph theory: the spectral rigidity of edge-weighted graphs under Laplacian reweighting, and the computation of semidefinite programming bounds for combinatorial optimization problems on highly regular graphs.

In the first part, we introduce and characterize the notion of *total conformal rigidity*, generalizing the concept of conformal rigidity due to Steinerberger and Thomas [70] from individual eigenvalues to sums of Laplacian eigenvalues. We show that total conformal rigidity is equivalent to *edge-rigidity*, the condition that every canonical spectral embedding of a graph onto a Laplacian eigenspace is edge-isometric. This equivalence is established via the analysis of semidefinite programs modeling sums of Laplacian eigenvalues, and is further connected to the Laplacian cospectrality of edges, the walk structure of signed line graphs, and the theory of gauge duality. We also show that edge-rigidity admits a complete structural characterization: a graph is edge-rigid if and only if it is 1-walk-regular or 1-walk-biregular. As combinatorial consequences of total conformal rigidity, we derive that the unweighted graph maximizes the number of spanning trees and minimizes the Kirchhoff index over all valid edge weightings. In the second part, we study semidefinite relaxations for the fractional cut-cover and MAX 2-SAT problems on graphs with high structural regularity. Using the theory of coherent configurations, association schemes, and gauge duality, we establish equality cases for a fundamental gauge duality inequality relating the MAXCUT SDP and its dual, and provide explicit closed-form expressions for optimal SDP solutions in terms of the eigenmatrices of the underlying association scheme.

The two parts of this thesis are united by a common algebraic thread: the theory of gauge duality. In the first part, gauge duality provides an intrinsic characterization of total conformal rigidity. In the second part, the same gauge-duality perspective recurs in the study of the MAXCUT SDP and its dual, where the fundamental inequality $\eta(G)\eta^{\circ}(G) \geq |E|$ becomes an equality precisely for graphs with sufficient regularity, including all totally conformally rigid graphs. Gauge duality thus serves as both a proof technique and a conceptual bridge, revealing that the spectral rigidity of a graph under Laplacian reweighting and the tightness of certain semidefinite bounds are intimately connected.

Keywords: Graph Theory, Algebraic Combinatorics, Semidefinite Optimization.

Contents

Introduction

The study of graphs through the lens of linear algebra is one of the most fruitful intersections in modern mathematics. The adjacency matrix, the Laplacian, and their spectral properties encode a surprising amount of combinatorial information about a graph, and conversely, structural properties of a graph impose strong algebraic constraints on these matrices. This thesis is situated at this intersection, drawing on tools from matrix analysis, semidefinite programming, and algebraic combinatorics to study two related families of problems.

The first theme of this thesis concerns the behavior of the Laplacian spectrum under edge reweighting. Given a graph $G = (V, E)$, one may ask how the eigenvalues of the weighted Laplacian $\mathcal{L}(w)$ vary as the edge weights w range over a simplex of valid assignments. A graph is said to be *conformally rigid* [70] if the uniform weight assignment simultaneously maximizes the second smallest and minimizes the largest Laplacian eigenvalue. Conformal rigidity is thus a statement about the spectral stability of a graph under reweighting, and its study connects naturally to questions in spectral geometry, graph partitioning, and spectral sparsification. In this thesis, we generalize conformal rigidity from individual eigenvalues to sums of eigenvalues, introducing the notion of *total conformal rigidity*. Our main result is that this global spectral stability condition is equivalent to a purely geometric property of the graph's spectral embeddings, which we call *edge-rigidity*: the requirement that every canonical embedding of the graph onto a Laplacian eigenspace maps adjacent vertices to points at equal distance. This equivalence is far from obvious, and its proof draws on the duality theory of semidefinite programs, the spectral theory of symmetric matrices, and the combinatorics of graph walks.

The second theme concerns semidefinite programming bounds for classical combinatorial optimization problems. The MAXCUT and MAX 2-SAT problems are among the most well-studied NP-hard problems in combinatorial optimization, and their classical approximation algorithms are based on semidefinite relaxations due to Goemans and Williamson [38]. When the underlying graph has a rich algebraic structure — such as belonging to an association scheme or a coherent configuration — the semidefinite programs simplify dramatically, and it becomes possible to compute optimal solutions in closed form. We exploit this to derive spectral bounds for the *fractional cut-cover* parameter and to compute the optimal value of the MAX 2-SAT semidefinite relaxation in terms of the eigenmatrices of the underlying association scheme.

The theory of gauge duality is what provides the conceptual link between these themes. In both cases, we study pairs of primal-dual monotone gauges parameterized by nonnegative edge weights, and look at the case when the constant vector weights provide equality cases to the Cauchy-Schwarz-type inequality associated with gauges. In Chapter 2, the gauges in question are the sums of the k-largest eigenvalues of the weighted Laplacian (and their corresponding duals), and edge-rigidity is what characterizes the equality cases discussed in 2.8. In Chapter 3, this same class of graphs provides a sufficient condition for the equality cases of the gauges η, η° associated with MAXCUT and FCC. This interplay between structural regularity of graphs and tightness in inequalities involving gauges will permeate the remainder of this work.

This thesis is thus organized as follows:

- **Chapter 1** collects the necessary background. We review the relevant linear algebra, including inner product spaces, adjoints, the spectral theorem, and positive semidefinite matrices. We then introduce the graph-theoretic objects of central importance: the adjacency and Laplacian matrices, coherent configurations and association schemes, distance-regular graphs, and walk-regular graphs. We conclude with an overview of linear and semidefinite programming duality and the theory of gauges.

- **Chapter 2** develops the theory of totally conformally rigid graphs. After introducing the relevant spectral embeddings and the notion of edge-rigidity in Section 2.2, we show in Section 2.3 that edge-rigidity is equivalent to pairwise Laplacian cospectrality of edges, and that all edge-transitive graphs are edge-rigid. In Section 2.4, we analyze the semidefinite programs for sums of Laplacian eigenvalues and prove that total conformal rigidity and edge-rigidity are equivalent. Section 2.5 provides a combinatorial characterization of edge-rigidity in terms of powers of the Laplacian matrix and signed line graphs, together with a polynomial-time recognition algorithm. Section 2.6 shows that all regular edge-rigid graphs are walk-regular and all biregular bipartite edge-rigid graphs are walk-biregular, leading to the complete characterization of edge-rigidity in terms of 1-walk-regularity and 1-walk-biregularity. Section 2.7 derives combinatorial consequences via majorization theory, and Section 2.8 interprets edge-rigidity through the lens of gauge duality. The results of this chapter were originally made available in [4], and are joint work with Gabriel Coutinho and Chris Godsil.

- **Chapter 3** studies semidefinite bounds for MAXCUT, the fractional cut-cover number, and MAX 2-SAT on highly regular graphs. Section 3.2.1 establishes that the fundamental gauge duality inequality $\eta(G)\eta^\circ(G) \geq |E|$ is an equality for graphs whose adjacency matrix belongs to or splits in a coherent configuration, with Section 3.2.2 extending this to totally conformally rigid graphs. Section 3.2.3 computes $\eta^\circ(G)$ in closed form for graphs in association schemes, yielding a spectral bound for FCC.

Section 3.3 treats the general quadratic programming framework and computes the optimal value of the MAX 2-SAT semidefinite relaxation for graphs in association schemes, as well as the dual gauge parameter for graphs arising from distance-regular graphs. The results of this chapter were originally published in [3], and are joint work with Gabriel Coutinho.

Chapter 1

Preliminaries

1.1 Linear algebra

Our main objects of study throughout this work will be matrices and vectors, and in this section we will give a brief overview of some of the core concepts required to fully understand the results from the next chapters. This section will be brief, and we refer the reader who wishes further details to [6].

We let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}_+$ be the set of nonnegative real numbers, and $\mathbb{R}_{++}$ be the set of positive real numbers. If X is a set with n elements, then we shall denote the vector space of real numbers indexed by X as $\mathbb{R}^X$ and the set of $n \times n$ real matrices with rows and columns indexed by X as $M_X(\mathbb{R})$. If we are not particularly interested in the elements of X per se, we shall simply write $\mathbb{R}^n$ and $M_n(\mathbb{R})$. This latter set of matrices is itself a vector space over $\mathbb{R}$, with dimension n^2, and it is also an algebra. This means that $M_n(\mathbb{R})$ is closed under scalar multiplication, and also under addition and multiplication. If V is a $\mathbb{R}$-vector space, and if there exists a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

such that $\langle v, w \rangle = \langle w, v \rangle$, $\langle \alpha v + \beta v', w \rangle = \alpha \langle v, w \rangle + \beta \langle v', w \rangle$, and $\langle v, v \rangle$ is positive if $v \neq 0$, for any $v, v', w \in V$, and $\alpha, \beta \in \mathbb{R}$, then we say that V is an *inner product space.* The most important example for us will be $V = \mathbb{R}^n$ with

$$\langle v, w \rangle := v^T w = \sum_{i=1}^{n} v_i w_i.$$

An inner product induces a norm $\|v\| := \langle v, v \rangle^{1/2}$, which in the previous example expands to

$$\|v\| = \sqrt{\sum_{i=1}^{n} v_i^2}.$$

The algebra $M_n(\mathbb{R})$ can also be endowed with an inner product, given by

$$\langle A, B \rangle := \mathrm{tr}(AB^T) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ij},$$

where $\operatorname{tr}(\cdot)$ denotes the *trace* of a matrix, i.e., the sum of its diagonal entries, and $(\cdot)^T$ denotes the transpose of a matrix. If $\operatorname{sum}(\cdot)$ is the operator that sums all entries of a given matrix, then

$$\langle A, B\rangle = \operatorname{tr}(AB^T) = \operatorname{sum}(A \circ B).$$

Here, $A \circ B$ is the entrywise product of matrices (known as the *Schur* product), i.e., $(A \circ B)_{ij} := A_{ij}B_{ij}$. We will use the symbol $\langle \cdot, \cdot\rangle$ for general inner products, and it will be clear from the context which explicit formulation is being referred to. The key concept related to inner product spaces that will appear throughout our work is the notion of the *adjoint* of an operator. Suppose V_1, V_2 are inner product spaces with inner products $\langle \cdot, \cdot\rangle_1, \langle \cdot, \cdot\rangle_2$, and let $\varphi : V_1 \to V_2$ be a linear map from V_1 to V_2. For the vector spaces we are interested in (real and finite dimensional), we can always find a unique linear map $\varphi^* : V_2 \to V_1$ called the *adjoint* of φ such that

$$\langle \varphi(v), w\rangle_2 = \langle v, \varphi^*(w)\rangle_1 \quad \text{for any } v \in V_1, w \in V_2.$$

Concretely, if we take $V_1 = V_2 = \mathbb{R}^n$ and $\varphi = A \in M_n(\mathbb{R})$, then $A^* = A^T$. A *self-adjoint* operator in this context, that is when $\varphi = \varphi^*$, is nothing more than a *symmetric* matrix. A useful example is related to the operators $\operatorname{Diag} : \mathbb{R}^n \to M_n(\mathbb{R})$ and $\operatorname{diag} : M_n(\mathbb{R}) \to \mathbb{R}^n$. The former maps a vector $v \in \mathbb{R}^n$ to a diagonal matrix $\operatorname{Diag}(v)$ such that $\operatorname{Diag}(v)_{ii} := v_i$. The latter maps a matrix $A \in M_n(\mathbb{R})$ to a vector $\operatorname{diag}(A)$ such that $\operatorname{diag}(A)_i := A_{ii}$. It is easy to see that $\operatorname{Diag}^* = \operatorname{diag}$.

The subspace of $M_n(\mathbb{R})$ formed by all symmetric matrices shall be denoted by $\mathbb{S}^n$, and it is straightforward to see that this space has dimension $\binom{n+1}{2}$. The most important fact about such matrices is that they can be orthonormally diagonalized, i.e., we can find an orthonormal basis of eigenvectors for $\mathbb{R}^n$. If $A \in \mathbb{S}^n$, then we can always find an orthogonal matrix U, where $U^TU = I = UU^T$, such that

$$A = U\Lambda U^T,$$

where Λ is a diagonal matrix with the eigenvalues of A. If $u_1, \dots, u_n$ are a set of orthonormal eigenvectors of A, with $Au_i = \lambda_i u_i$, then

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^T.$$

If A has r distinct eigenvalues $\lambda_1, \dots, \lambda_r$, then we can add all terms $u_i u_i^T$ corresponding to a given eigenvalue into a matrix E_i, such that

$$A = \sum_{i=1}^{r} \lambda_i E_i.$$

The matrices E_i are orthogonal projectors onto the eigenspace of λ_i, that is, $I = \sum_i E_i$, $E_i^T = E_i$, $E_iE_j = \delta_{ij}E_i$, where δ_{ij} is the *Kronecker delta* that equals zero if $i \neq j$ and 1 otherwise. A more self-contained treatise on these facts and how to prove them can be found in [35, Ch. 8].

The matrix A may not be invertible, but we can define its Moore–Penrose *pseudo-inverse* $A^\dagger$, which is given by

$$A^\dagger := \sum_{i:\lambda_i \neq 0} \frac{1}{\lambda_i} E_i.$$

The product $AA^\dagger$ will be equal to a projection matrix onto the column space of A, with rank equal to the rank of A.

To finish this section, we give a brief overview of *positive semidefinite* (PSD) matrices. A matrix $A \in \mathbb{S}^n$ is PSD if all of its eigenvalues are nonnegative. This is equivalent to requiring that $v^T A v \geq 0$ for all $v \in \mathbb{R}^n$, or to requiring that $A = BB^T$ for some matrix B. Since all eigenvalues of A are nonnegative, we can write

$$A = U\Lambda U^T = (U\Lambda^{1/2}U^T)(U\Lambda^{1/2}U^T)^T,$$

where $\Lambda^{1/2}$ is the diagonal matrix with the square roots of the eigenvalues of A on its diagonal entries. We then let $A^{1/2} := U\Lambda^{1/2}U^T$ denote the *square root* of A. The set of PSD matrices will be denoted by $\mathbb{S}^n_+$, and we note that, although this is not a subspace, it is a *convex cone*, that is, it is closed under nonnegative scalar multiplication, and it is closed under convex combinations of its elements. If $A, B \in \mathbb{S}^n$, then we write $A \succeq B$ if $A - B \in \mathbb{S}^n_+$. It is easy to see that if A is PSD, then for any matrix B, it follows that BAB^T is also PSD. As a consequence of this, we get the following useful fact:

Proposition 1.1.1. If $A, B \in \mathbb{S}^n_+$ are positive semidefinite matrices such that $\langle A, B\rangle = 0$, then $AB = 0$.

Proof. Let $A^{1/2}$ be the PSD square root of A, and write

$$0 = \langle A, B\rangle = \operatorname{tr}(AB) = \operatorname{tr}(A^{1/2}BA^{1/2}).$$

As $B \succeq 0$, it follows that $A^{1/2}BA^{1/2} \succeq 0$. The trace of a PSD matrix is only zero when the matrix itself is zero, thus $A^{1/2}BA^{1/2} = 0$. In particular, for any vector $v \in \mathbb{R}^n$, it follows that

$$v^T A^{1/2}BA^{1/2}v = (A^{1/2}v)^T B(A^{1/2}v) = 0,$$

and as B is PSD, this implies that $B(A^{1/2}v) = 0$. Since v was arbitrary, this shows that $BA^{1/2} = 0$, which implies that $BA = 0$. As $AB = (BA)^T$, this shows that $AB = 0$, concluding the proof. ■

1.2 Graphs and regularity

The matrices and vectors discussed in the previous section will be algebraic tools for us to better understand properties of combinatorial objects, namely *graphs*. In this work, a graph $G = (V, E)$ consists of a set of vertices V, and a set of edges E. All graphs in this work will be finite, meaning $|V| < \infty$, *simple*, meaning *undirected*, *loopless* and without multiple edges, and *connected*. For a comprehensive treatise on graph theory, we refer the reader to [26]. There are two matrices associated with graphs that will be of interest to us: the *adjacency matrix* and the *Laplacian matrix*. The adjacency matrix $A := A(G) \in \mathbb{S}^V$ is a symmetric matrix where $A_{ab} = 1$ if ab is an edge, and $A_{ab} = 0$ otherwise. The *unweighted* Laplacian matrix L is the matrix given by

$$L = \sum_{ab \in E} (e_a - e_b)(e_a - e_b)^T,$$

where $e_a \in \mathbb{R}^V$ is the canonical vector indexed by a. From this definition, it is easy to see that $L \in \mathbb{S}^V_+$. The diagonal entries of L store the degrees of the vertices of G, and its off-diagonal entries are either -1 if they correspond to an edge, or 0 otherwise. As a consequence of G being connected, it follows that L has rank $n-1$, i.e., only one of its eigenvalues is equal to zero. The corresponding eigenvector of this zero eigenvalue is the all-ones vector $\mathbb{1}$, with eigenprojector given by $(1/n)J$, where $J = \mathbb{1}\mathbb{1}^T$ is the all-ones matrix. We can also define the Laplacian matrix w.r.t. edge weights $w \in \mathbb{R}^E$. For an edge $ab \in E$, we define

$$z_{ab} := e_a - e_b,$$

and we also let $L_{ab} := z_{ab} z_{ab}^T$. The *weighted Laplacian* is the linear operator $\mathcal{L} : \mathbb{R}^E \to \mathbb{S}^V$ such that

$$\mathcal{L}(w) := \sum_{ab \in E} w_{ab} L_{ab} = \sum_{ab \in E} w_{ab} z_{ab} z_{ab}^T, \tag{1.1}$$

with $L = \mathcal{L}(\mathbb{1})$. If we restrict ourselves to the set of nonnegative edge weights $\mathbb{R}^E_+$, then the image of $\mathcal{L}$ is contained in the cone of positive semidefinite matrices $\mathbb{S}^V_+$. The adjoint $\mathcal{L}^* : \mathbb{S}^V \to \mathbb{R}^E$ of $\mathcal{L}$ can be derived from the following relation:

$$\langle \mathcal{L}(w), X \rangle = \langle w, \mathcal{L}^*(X) \rangle \quad \text{for any } w \in \mathbb{R}^E,\ X \in \mathbb{S}^V.$$

By expanding this definition, we obtain

$$\mathcal{L}^*(X)_{ab} = z_{ab}^T X z_{ab} = X_{aa} + X_{bb} - 2X_{ab} \quad \text{for any } ab \in E. \tag{1.2}$$

1.2.1 Coherent configurations and association schemes

In this section, we briefly discuss the concepts of coherent configurations and association schemes. For a standard reference, see [8], and for further information on the algebras associated with these objects, see [2]. We say that a set $\mathcal{C} = \{A_0, A_1, \ldots, A_d\}$ of square matrices with entries in $\{0, 1\}$ is a *coherent configuration* if the following hold:

(1) $\sum_i A_i = J$;

(2) If $A_i \in \mathcal{C}$, then $A_i^T \in \mathcal{C}$;

(3) If $A_i \in \mathcal{C}$ and A_i has a nonzero diagonal entry, then it is a diagonal matrix;

(4) There are nonnegative integers p_{ij}^{ℓ} such that

$$A_i A_j = \sum_{\ell=0}^{d} p_{ij}^{\ell} \cdot A_l,$$

for any indices $0 \leq i, j \leq d$.

The diagonal matrices in $\mathcal{C}$ that partition the identity matrix I are called *fibers*, and if there is only one of these, that is, if $I \in \mathcal{C}$, then we say that the configuration is *homogeneous*. If all matrices in $\mathcal{C}$ commute, we say that the configuration is *commutative*, and in this case it is easy to see that the configuration must also be homogeneous: the only nonzero diagonal matrix with entries in $\{0, 1\}$ that commutes with J is the identity matrix. If all matrices in $\mathcal{C}$ are symmetric, then by taking the transpose of the resulting expression for $A_i A_j$ given by item (4), we see that $\mathcal{C}$ must also be commutative, and hence homogeneous. For the remainder of this text, we shall refer to symmetric coherent configurations as *association schemes*, following Cameron [18]. We shall also refer to coherent configurations and association schemes simply as configurations and schemes, respectively.

Coherent configurations were initially introduced by Higman [45] to study permutation groups, as, for instance, one can always construct a coherent configuration from the orbitals of a permutation group acting on some set, as well as from the Cayley graphs generated by its conjugacy classes. Association schemes on the other hand were first introduced to study statistical experiments, but have since found many important applications in graph theory — e.g. in the study of distance-regular and strongly-regular graphs [17] —, coding theory — e.g. in the study of error-correcting codes [25] —, and in many other areas of combinatorics [8]. In the case of commutative configurations, we get that the linear span of the matrices in $\mathcal{C}$ over $\mathbb{C}$ forms a commutative $*$-algebra — that is, an algebra closed under conjugate transposition —, which in turn implies that there exists a basis of primitive idempotents $E_0 = (1/|V|)J, E_1, \ldots, E_d$ that are the projectors onto the common eigenspaces of the matrices in $\mathcal{C}$ (see for instance [8, Ch. 2]). The $*$-algebra spanned by the matrices in a configuration $\mathcal{C}$ is known as a *coherent algebra*, and in the case of association schemes we obtain the well-known Bose–Mesner algebra. The familiar vertex-transitive and edge-transitive graphs are intimately related to coherent algebras. If $\Gamma = \mathrm{Aut}(G)$ denotes the automorphism group of a graph G, then the set

$$C(\Gamma) := \{M \in M_V(\mathbb{C}) : MP = PM,\ \forall P \in \Gamma\},$$

known as the centralizer algebra (or commutant algebra) of Γ, is a coherent algebra, where the group Γ is identified with its natural representation as a subgroup of permutation matrices. Some structural properties of G correspond to algebraic properties of $C(\Gamma)$: if G is vertex-transitive then $C(\Gamma)$ is always homogeneous; and if G is edge-transitive we have three possibilities: (i) if G is 1-arc-transitive, then $C(\Gamma)$ will be homogeneous and A will belong to $\mathcal{C}$; (ii) if G is edge-transitive and vertex-transitive but not 1-arc-transitive, then $C(\Gamma)$ will be homogeneous and A will *split* in $\mathcal{C}$, that is, $A = A_i + A_i^T$ for some index $0 \le i \le d$; or (iii) if G is edge-transitive but not vertex-transitive, then it will be bipartite and thus $C(\Gamma)$ will have two fibers and A will split in $\mathcal{C}$ (see [14, Ch. 17] for more details).

1.2.2 Distance-regular graphs

The main examples of association schemes are the so-called *distance-regular graphs*, and much of the theory about configurations and schemes arises from generalizations of properties exhibited by these graphs. To introduce the subject, we start with an example. If we let $[n] := \{1, \dots, n\}$, fix some integer $1 \le d \le n$ and define V as the set of all subsets of $[n]$ of size d, we can construct the so-called *Johnson graph* $J(n, d)$ by connecting two vertices if their intersection has size $d-1$. Two vertices of $J(n, d)$ are at distance i if, and only if, their intersection has size $d-i$, and so we can consider the set $\{I, A_1, A_2, \dots, A_d\}$ where A_i is the adjacency matrix connecting vertices at distance i, with $A_1 = A$ the usual adjacency matrix. The graphs corresponding to A_i are usually called the *generalized Johnson graphs* $J(n, d, i)$, and the graph $K(n, d) := J(n, d, d)$ that connects subsets if they are disjoint is also known as the *Kneser graph*. It can be shown that this set of matrices is an association scheme, which will motivate the definition below.

Definition 1.2.1. If G is a connected k-regular graph with diameter d and adjacency matrix A, then we say that G is *distance-regular* if the matrices $\{I, A_1 = A, A_2, \dots, A_d\}$ form an association scheme.

Let us now study the combinatorial implications of this definition. First, we note that the parameters $k_i := p_{ii}^0$ of the scheme are precisely the degrees of the graphs induced

(a) Petersen graph constructed as $K(5,2)$. (b) Johnson graph $J(5,2)$.

Fig. 1.1: The Petersen and Johnson graphs arising from the Johnson scheme with parameters $n = 5, d = 2$.

by the matrices A_i, i.e., the graph of A_i is k_i-regular. We also note that since the distance in a graph is a metric, we can use the triangle inequality to obtain that if $p^{\ell}_{ij} \neq 0$, then $\ell \leq i + j$, $j \leq i + \ell$, and $i \leq j + \ell$, and in particular, $p^{i}_{j1} = 0$ if $j \notin \{i-1, i, i+1\}$. With this, we can define the main parameters for a distance-regular graph:

- $c_i := p^{i}_{(i-1)1}$: given vertices u, v at distance i, c_i is the number of neighbors of v that are at distance $i - 1$ from u, i.e., closer to u;
- $b_i := p^{i}_{(i+1)1}$: b_i is the number of neighbors of v that are at distance $i + 1$ from u, i.e., farther from u;
- $a_i := k - b_i - c_i$: a_i is the number of neighbors of v at the same distance i from u.

If we fix any vertex u and partition the other vertices of the graph according to their distance from u, we obtain the following diagram:

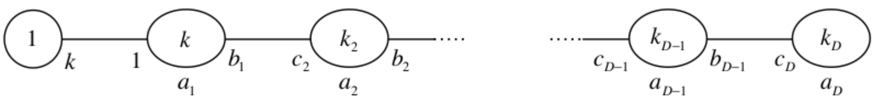

Fig. 1.2: Distance diagram [72, Figure 1].

It is always possible to perform this type of partition in any connected graph, but the fact that the graph is distance-regular shows us that the parameters of this diagram are the same regardless of which vertex u we fix. In fact, this is one possible definition of distance-regular graphs: if the parameters of the distance partition are the same regardless of the choice of the fixed vertex, then we say the graph is distance-regular. With this, we can count the number of edges between two consecutive components i and $i + 1$ of the partition in two ways: there are $k_i b_i$ edges going from component i to $i + 1$, and equally

$k_{i+1}c_{i+1}$ edges going from component $i+1$ to i, so we obtain

$$k_{i+1} = \frac{k_i b_i}{c_{i+1}},$$

and naturally $n = 1 + k_1 + \cdots + k_d$. We will denote by $N_i(u)$ the set of vertices at distance i from u (i.e., adjacent to u in the graph of A_i). With this, we can show the following characterization of distance-regular graphs:

Theorem 1.2.2. Let G be a k-regular connected graph with diameter d and adjacency matrix A. Then G is distance-regular if, and only if, there exist nonnegative integers k_i, b_i, c_i such that $|N_i(v)| = k_i$ for any $v \in V$, and for any $u \in N_i(v)$, it holds that

$$|N_1(u) \cap N_{i-1}(v)| = c_i,$$
$$|N_1(u) \cap N_{i+1}(v)| = b_i.$$

Proof. The forward direction is immediate from the previous considerations, so assume there exist integers k_i, b_i, c_i satisfying the stated properties, and define $a_i := k - b_i - c_i$. We now want to show that the matrices $I, A, A_2, \ldots, A_d$ form a symmetric association scheme, and for this, it is enough to check that condition (4) of the definition holds. It will be useful to define $A_{-1} := A_{d+1} = 0$ and leave c_{d+1}, b_{-1} free (their values will not be relevant). With this, we note that

$$(AA_i)_{uv} = |N_1(u) \cap N_i(v)| = \begin{cases} b_{i-1}, & \text{if } u \in N_{i-1}(v), \\ a_i, & \text{if } u \in N_i(v), \\ c_{i+1}, & \text{if } u \in N_{i+1}(v), \end{cases}$$

that is, for any $i \in \{0, \ldots, d\}$, it holds that

$$AA_i = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1}.$$

We can apply induction on i to show that A_i is always a polynomial in A of degree i, and also that $A^i \in \operatorname{span}(\{I, A, A_2, \ldots, A_i\})$. The cases $i = 0$ and $i = 1$ are immediate. For $i = 2$, we have

$$A^2 = b_0 I + a_1 A + c_2 A_2,$$

so A^2 is a linear combination of I, A, A_2, and since $c_2 \neq 0$ (because G is connected), we have that A_2 is a polynomial $\nu_2(A)$ of degree 2 in A, given by

$$c_2 A_2 = c_2 \nu_2(A) = A^2 - a_1 A - b_0 I.$$

If the result holds for i, then it follows that

$$c_{i+1}A_{i+1} = c_{i+1}\nu_{i+1}(A) = A\nu_i(A) - a_i\nu_i(A) - b_{i-1}\nu_{i-1}(A),$$

so A_{i+1} is a polynomial of degree $i+1$ in A. Now, writing $A^i = \sum_{\ell=0}^{i} \alpha_\ell A_\ell$ with $\alpha_l \in \mathbb{R}$, we get

$$A^{i+1} = \sum_{\ell=0}^{i} \alpha_\ell A_\ell A,$$

and using that $A_\ell A$ is a linear combination of $A_{\ell-1}, A_\ell, A_{\ell+1}$, it follows that $A^{i+1} \in \operatorname{span}(\{I, A, A_2, \ldots, A_{i+1}\})$, which concludes the induction. This shows that $A_i A_j$ is a polynomial in A, and since

$$AA_d = b_{d-1}A_{d-1} + a_d A_d,$$

it follows that this polynomial has degree at most d, so $A_i A_j \in \operatorname{span}(\{I, A, A_2, \ldots, A_d\})$, as required. ■

This result shows that a distance-regular graph G is completely determined by the parameters k_i, b_i, c_i, so we can define the *intersection array* of G as

$$\iota(G) := \{b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d\}.$$

In general, given an intersection array, it is not an easy task to determine whether there exists distance-regular graphs with these parameters, nor is it easy to determine whether a distance-regular graph is the only one with a given array (up to isomorphism). The Petersen graph has intersection array $\{3, 2; 1, 1\}$, and it is possible to prove that it is uniquely determined by this array, that is, any distance-regular graph with this array is isomorphic to the Petersen graph. The smallest intersection array corresponding to a pair of non-isomorphic distance-regular graphs is $\{6, 3; 1, 2\}$, as both the Hamming graph $H(2, 4)$ and the Shrikhande graph share exactly this array (for more details, we refer the reader to [72, Ch. 2]).

1.2.3 Walk-regular graphs

In this section, we briefly discuss a family of regular graphs that will play a central role in the next chapters. A graph G with adjacency matrix A is said to be *walk-regular* if, for any nonnegative integer ℓ, the number of closed walks of length ℓ starting at any vertex is constant, i.e., A^ℓ has constant diagonal. From this definition, we can easily see that both vertex-transitive and distance-regular graphs are walk-regular, and also that, more generally, any graph whose adjacency matrix belongs to a homogeneous coherent algebra is walk-regular. These graphs were originally introduced by Godsil and McKay [34], who also notably proved that they are precisely the graphs for which all vertices are pairwise cospectral, that is, the adjacency matrices of $G - u$ and $G - v$ have the same characteristic polynomial for all vertices $u, v \in V$. Rowlinson [64] found a way of characterizing distance-regular graphs in terms of walks, showing that such graphs are defined by the property that the number of walks between any two vertices is a constant that depends only on the length of the walk and the distance between the vertices. Equipped with the previous definitions, one can see walk-regularity and distance-regularity as particular instances of k-walk-regularity, introduced by Dalfó, Fiol and Garriga [22].

Definition 1.2.3. We say that a connected graph G of diameter d is *k-walk-regular*, with $0 \le k \le d$, if the number of walks of length ℓ between any two vertices at distance i is a constant depending only on ℓ and i, for all integers $\ell \ge 0$ and $0 \le i \le k$.

From this definition, it follows that 0-walk-regular graphs are precisely the class of walk-regular graphs, and d-walk-regular graphs are the class of distance-regular graphs.

1.3 Linear and semidefinite programs

In this section, we will briefly discuss the concepts of linear and semidefinite programs, and the most important results related to such problems in the context of this work. A *linear program* (LP) is the problem of optimizing a linear functional over a subset of euclidean space defined by a finite number of affine inequalities. We first define the following primal LP:

$$\max\{c^T x : Ax \le b,\ x \in \mathbb{R}^n_+\}, \tag{1.3}$$

where $A \in M_{m\times n}(\mathbb{R})$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. The matrix A and the vector b define the constraints of the LP, and if $x \in \mathbb{R}^n_+$ and $Ax \le b$, then we say that x is a *feasible* solution to (1.3). The vector c defines the linear functional we aim to maximize. An LP which has a feasible solution is also called feasible, and if x is feasible and $c^T x \ge c^T x'$ for any other feasible x', we say that x is an *optimal* solution. Not all LPs have feasible solutions, and optima may not be unique. The key concept in the theory of LPs is the notion of *duality*. We can define the *dual* LP of (1.3) as

$$\min\{b^T y : A^T y \ge c,\ y \in \mathbb{R}^m_+\}, \tag{1.4}$$

and note that

$$c^T x \le y^T A x \le b^T y,$$

meaning that the objective value of any feasible solution to (1.3) is always bounded above by the objective value of any feasible solution to (1.4). This fact is usually referred to as *weak duality*. A much stronger result relating these two LPs is known as *strong duality*:

Theorem 1.3.1. If (1.3) and (1.4) are both feasible, then any primal-dual pair of optimal solutions $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ has the same objective value, i.e.,

$$c^T x = b^T y.$$

■

The proof of this result, although highly nontrivial, is standard among linear programming texts, and so we omit it here (see, for instance, [12]). An important consequence of strong duality are the *complementary slackness conditions*:

Corollary 1.3.2. Given a pair of feasible solutions $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ to (1.3) and (1.4), respectively, then they are optimal with equal objective values if, and only if,

$$x^T(A^T y - c) = 0 \quad \text{and} \quad y^T(b - Ax) = 0.$$

In particular, if an entry of a primal (resp. dual) variable is nonzero, then the corresponding dual (resp. primal) inequality must be tight. ■

We now turn our attention to *semidefinite programs* (SDPs), which can be roughly described as problems of optimizing a linear functional over a finite number of affine constraints on a positive semidefinite matrix variable. The theory of SDPs is rich and has led to many important results in combinatorics and computer science. Our goal here is to describe how to formulate SDPs and their duals, and to state the analogues of strong duality and complementary slackness in this new context. We define a primal SDP as:

$$\max\{\langle C, X\rangle : \langle A_i, X\rangle \leq b_i \text{ for } i \in [m],\ X \succeq 0\}, \tag{1.5}$$

where $C, A_i \in \mathbb{S}^n$ and $b \in \mathbb{R}^m$. The corresponding dual problem is given by

$$\min\left\{b^T y : \sum_{i=1}^m y_i A_i \succeq C,\ y \in \mathbb{R}^m_+\right\}. \tag{1.6}$$

As before, we can easily verify weak duality:

$$\langle C, X\rangle \leq \sum_{i=1}^m y_i \langle A_i, X\rangle \leq b^T y.$$

Unlike the LP case, a pair of feasible primal-dual SDPs need not attain the same optimal value. To rule this out, one imposes the existence of a *Slater point*: a strictly feasible solution. Concretely, a primal Slater point is a matrix $X \succ 0$ (i.e., positive definite) such that $\langle A_i, X\rangle < b_i$ for all $i \in [m]$, and a dual Slater point is a vector $y \in \mathbb{R}^m_{++}$ such that $\sum_{i=1}^m y_i A_i \succ C$. Strong duality for SDPs then reads as follows:

Theorem 1.3.3. If the primal (1.5) has a Slater point and is bounded above, then the dual (1.6) attains its infimum and both problems have the same optimal value. Symmetrically, if the dual (1.6) has a Slater point and is bounded below, then the primal (1.5) attains its supremum and both problems have the same optimal value. If both problems have Slater points, then both optima are attained and equal. ■

Again, the proof of this result is highly nontrivial but standard in the SDP literature (see, for instance, [9, 15]). When both optima are attained and equal, the optimal solutions X and y satisfy the following *complementary slackness conditions*:

Corollary 1.3.4. Given a pair of feasible solutions X to (1.5) and y to (1.6), they are optimal with equal objective values if, and only if,

$$\left(\sum_{i=1}^{m} y_i A_i - C\right) X = 0 \quad \text{and} \quad y_i(b_i - \langle A_i, X\rangle) = 0 \text{ for all } i \in [m].$$

■

1.4 Gauges

In this section, we briefly discuss *gauges*, which are certain functions that will serve as the conceptual link between the later chapters of this work. The standard reference in a more general context is Rockafellar [63], and a comprehensive treatment in the context of graph theory can be found in [10]. A function $g : \mathbb{R}^E_+ \to \mathbb{R}$ is called a *gauge* if

(a) g is *nonnegative*, meaning that $g(w) \geq 0$ for all $w \in \mathbb{R}^E_+$, and $g(0) = 0$;

(b) g is *positively homogeneous*, meaning that $g(\alpha w) = \alpha g(w)$ for all $\alpha \geq 0$ and $w \in \mathbb{R}^E_+$;

(c) g is *subadditive*, meaning that $g(w + w') \leq g(w) + g(w')$ for all $w, w' \in \mathbb{R}^E_+$.

If in addition $g(w) = 0$ implies $w = 0$, then g is *positive definite*, and if $g(w) \leq g(w')$ whenever $w \leq w'$ coordinate-wise, then we say the gauge is *monotone*. Positive definite monotone gauges are restrictions of norms to the nonnegative orthant.

A positive definite monotone gauge g admits a *dual gauge*, defined as

$$g^\circ(w) = \max_{y \in \mathbb{R}^E_+}\{w^T y : g(y) \leq 1\} = \min_{\lambda \geq 0}\{\lambda : w^T y \leq \lambda g(y) \text{ for all } y \in \mathbb{R}^E_+\}.$$

As an immediate consequence, we obtain a type of Cauchy–Schwarz inequality: for all $w, y \in \mathbb{R}^E_+$,

$$w^T y \leq g(w) g^\circ(y). \tag{1.7}$$

Standard theory of convex duality implies that g° is also a positive definite monotone gauge, and that the dual of the dual gauge is the original gauge, i.e., $g^{\circ\circ} = g$. It is often very useful to characterize the convex corners that play the role of unit balls of a gauge, that is, the sets of the form $\{w \in \mathbb{R}^E_+ : g(w) \leq 1\}$ for some gauge g. If B is the unit convex corner associated to a gauge g and D is the convex corner associated to its dual gauge g°, then a standard presentation of the gauges is given as the Minkowski functionals of these convex corners:

$$g(w) = \min_{\lambda \geq 0} \{\lambda : w \in \lambda B\}, \quad g^{\circ}(w) = \min_{\lambda \geq 0} \{\lambda : w \in \lambda D\}.$$

We also have the following duality relationship:

$$g(w) = \max_{y \in D} w^T y \quad \text{and} \quad g^{\circ}(w) = \max_{y \in B} w^T y.$$

We will see in the upcoming chapters that many famous graph parameters can be studied within the framework of gauge duality, which in turn allows us to further understand the geometry associated with solutions to certain optimization problems.

Chapter 2

Total Conformal Rigidity

In this chapter, we introduce the notion of totally conformally rigid graphs, and show that it can be characterized in a number of different ways, connecting with concepts from algebraic and spectral graph theory, semidefinite optimization and matrix analysis. The contents of this chapter were originally made available in [4], and are joint work with Gabriel Coutinho and Chris Godsil.

2.1 Overview

Let $G = (V, E)$ be a simple, connected, undirected graph on n vertices with at least one edge, and let $A := A(G)$ denote its adjacency matrix. In this chapter, we will be interested in optimization problems for the eigenvalues of the weighted Laplacian $\mathcal{L}(w)$ with respect to normalized edge weights w. The simplex of valid edge weights is thus defined as

$$\Delta_E := \{w \in \mathbb{R}^E : w \geq 0,\ \mathbb{1}^T w = |E|\}.$$

We shall order the eigenvalues of $\mathcal{L}(w)$ as

$$0 = \lambda_1(w) \leq \lambda_2(w) \leq \ldots \leq \lambda_n(w).$$

For any $w \in \Delta_E$, it follows that $\lambda_1(w) = 0$, since $\mathcal{L}(w)\mathbb{1} = 0$. The rank of $\mathcal{L}(w)$ is thus at most $n - 1$ (with equality when $w = \mathbb{1}$), but it may be smaller since w is allowed to have entries equal to zero. By the *nontrivial* eigenvalues of $\mathcal{L}(w)$, we shall mean $\lambda_2(w), \ldots, \lambda_n(w)$.

In [70], Steinerberger and Thomas studied the concept of *conformal rigidity* of a graph, which can be defined as follows:

Definition 2.1.1. We say that a graph G is *conformally rigid* if for any $w, w' \in \Delta_E$, we have

$$\lambda_2(w) \leq \lambda_2(\mathbb{1}) \quad \text{and} \quad \lambda_n(\mathbb{1}) \leq \lambda_n(w'),$$

that is, if $\mathbb{1}$ maximizes the second smallest eigenvalue of $\mathcal{L}(w)$, and minimizes its largest eigenvalue over all valid edge weights. If the former holds, we say the graph is *lower conformally rigid*, and if the latter holds, we say it is *upper conformally rigid*.

The terminology is inspired by conformal geometry: just as a conformal map distorts distances while preserving angles, a reweighting of edges distorts the spectral geometry of the graph while leaving its combinatorial structure intact. A conformally rigid graph resists such distortion — no reweighting can push the spectrum beyond its unweighted range. Conformal rigidity can be characterized in terms of spectral embeddings onto the eigenspaces of $\lambda_2(\mathbb{1})$ and $\lambda_n(\mathbb{1})$. The authors of [70] also show that distance-regular graphs are conformally rigid, and provide sufficient conditions for Cayley graphs to exhibit conformal rigidity. In a follow-up paper, Gouveia, Steinerberger and Thomas [39] further studied necessary and sufficient conditions for Cayley graphs and vertex-transitive graphs to be conformally rigid.

In this work, we generalize conformal rigidity to sums of eigenvalues. More precisely, we define the following notion:

Definition 2.1.2. Let $S_k(w)$ and $s_k(w)$ denote the sum of the k largest and k smallest nontrivial eigenvalues of $\mathcal{L}(w)$, respectively, for some $k \in [n-1]$. We say that G is *k-conformally rigid* if, for any $w, w' \in \Delta_E$,

$$s_k(w) \leq s_k(\mathbb{1}) \quad \text{and} \quad S_k(\mathbb{1}) \leq S_k(w').$$

If $\mathbb{1}$ minimizes S_k over Δ_E, we say that G is *upper k-conformally rigid*, and if it maximizes s_k, we say it is *lower k-conformally rigid*. If G is k-conformally rigid for all $k \in [n-1]$, we say that G is *totally conformally rigid*.

The notion of conformal rigidity introduced in [70] is equivalent to our notion of 1-conformal rigidity.

Laplacian eigenvalues arise naturally in several areas of graph theory and combinatorial optimization. Ky Fan's classical result [27] characterizes sums of these eigenvalues variationally, and they appear prominently in the study of isoperimetric constants [56] and graph partitioning [57]. More recently, sums of Laplacian eigenvalues have been studied in connection with the *spectral width* $\lambda_n(w) - \lambda_2(w)$, whose minimization over edge weights is itself a semidefinite program related to uniform sparsest cuts [70]. A well-known open problem involving these sums is *Brouwer's conjecture* [43], which asserts that, for any simple undirected graph G, $S_k(\mathbb{1}) \leq |E| + \binom{k+1}{2}$ for any $k \in [n-1]$. The conjecture has been verified for many classes of graphs but remains open in general.

A key notion in our work is that of *edge-rigidity*—the condition that every canonical spectral embedding of G onto a Laplacian eigenspace is edge-isometric. As we will show, if $L = \sum_{i=1}^{r} \lambda_i E_i$ is the spectral decomposition of L, then edge-rigidity is equivalent to the condition that $\mathcal{L}^*(E_i) = \gamma_i \mathbb{1}$ for some constant $\gamma_i > 0$ for every $i \in \{2, \dots, r\}$. Therefore, a consequence of edge-rigidity is that $\mathcal{L}^*(L^\dagger)$ is a constant vector, where $L^\dagger$ is the Moore-Penrose pseudoinverse of L, leading to a vast list of derived properties satisfied by edge-rigid graphs, for instance:

- The effective resistance of any edge is the same, and equal to $(|V| - 1)/|E|$.
- Every edge has the same commute time from one of its endpoints to the other.
- The number of spanning trees containing any given edge is the same (that is, the graph is equiarboreal).

The fact that the last condition holds for 1-walk-regular graphs is due to Godsil [32], and is a consequence of the fact that 1-walk-regular graphs are edge-rigid, as we will see in Section 2.6.

2.2 Edge-rigidity and spectral embeddings

In this section, we will define the notion of edge-rigidity in terms of spectral embeddings of graphs. These were originally introduced by Hall [44] to study the problem of placing n connected points in Euclidean space, and have since become a prominent technique with applications in many different areas of computer science and mathematics, such as graph clustering [55, 29] and data mining [73, 65].

Let $\lambda > 0$ be an eigenvalue of L with multiplicity m, and let $\mathcal{E}_\lambda \cong \mathbb{R}^m$ be its corresponding eigenspace.

Definition 2.2.1. Let $U \in \mathbb{R}^{n \times m}$ be a matrix whose columns form an orthonormal basis for $\mathcal{E}_\lambda$. The collection of vectors $\mathcal{U} = \{u_1, \dots, u_n\} \subset \mathbb{R}^m$, where $u_i = U^T e_i$ is the i-th row of U, is called the *canonical spectral embedding* of G onto $\mathcal{E}_\lambda$.

These embeddings provide a geometric realization of G in $\mathbb{R}^m$. Because the columns of U are eigenvectors corresponding to $\lambda > 0$, they are orthogonal to $\mathbb{1}$. Consequently, the embedding is centered at the origin, meaning $\sum_{i=1}^n u_i = 0$. A particularly structured class of embeddings consists of those where all adjacent vertices are separated by the same Euclidean distance.

Definition 2.2.2. A canonical spectral embedding $\mathcal{U} = \{u_1, \dots, u_n\}$ onto $\mathcal{E}_\lambda$ is *edge-isometric* if there exists a constant $\gamma > 0$ such that $\|u_a - u_b\|^2 = \gamma$ for all $ab \in E$.

The existence of an edge-isometric embedding places severe algebraic constraints on the graph. Let $X := UU^T$ be the orthogonal projector onto $\mathcal{E}_\lambda$. Note that X is independent of the choice of orthonormal basis U: different bases yield geometrically congruent embeddings related by an orthogonal transformation of $\mathbb{R}^m$, and the edge-isometric property is therefore an intrinsic property of the eigenspace $\mathcal{E}_\lambda$ rather than of any particular basis. The squared distance between adjacent vertices a and b is given by

$$\|u_a - u_b\|^2 = \|u_a\|^2 + \|u_b\|^2 - 2u_a^T u_b = X_{aa} + X_{bb} - 2X_{ab} = \mathcal{L}^*(X)_{ab},$$

hence the canonical embedding onto $\mathcal{E}_\lambda$ is edge-isometric if and only if $\mathcal{L}^*(X) = \gamma\mathbb{1}$ for some constant $\gamma > 0$.

The notion of conformal rigidity studied in [70] requires that the embeddings onto $\mathcal{E}_{\lambda_2}$ and $\mathcal{E}_{\lambda_n}$ alone be edge-isometric. We are interested in the stronger condition that *every* canonical embedding is edge-isometric, capturing a global spectral regularity of the graph:

Definition 2.2.3. We say that a graph G is *edge-rigid* if all canonical embeddings onto the Laplacian eigenspaces are edge-isometric, that is, for every eigenprojector E_i of L corresponding to a nonzero eigenvalue there exists $\gamma_i > 0$ such that $\mathcal{L}^*(E_i) = \gamma_i\mathbb{1}$.

Before proceeding, we briefly remark on the embedding induced by the trivial eigenvalue $\lambda_1(\mathbb{1}) = 0$. Its eigenprojector is $E_1 := (1/n)\mathbb{1}\mathbb{1}^T$, and thus every vertex would be mapped into a single vector, resulting in an embedding which is trivially edge-isometric, with $\mathcal{L}^*(E_1) = 0$. Since this is true for any connected graph, we omit it from the definitions above.

2.3 Laplacian cospectrality

Recall that two graphs G and H are called *cospectral* if their adjacency matrices share the same characteristic polynomial. The study of cospectral graphs was started in the 1950s [41], and has since become a central topic in spectral graph theory [68, 33, 23]. Recently, Godsil, Sun and Zhang [36] showed how to use 1-walk-regularity to obtain families of non-isomorphic cospectral graphs via edge deletion. In this section, we shall similarly define a notion of cospectrality for edges that can be used to characterize edge-rigidity. In a later section, we shall see that these notions can also be characterized in terms of 1-walk-regular graphs.

Given an edge $ab \in E$, the matrix $L - L_{ab}$ is the Laplacian of the graph $G - ab$ obtained by removing the edge ab while retaining the vertices a and b, so we define:

Definition 2.3.1. Two edges $ab, cd \in E$ are called *Laplacian-cospectral* if the matrices

$$L - L_{ab} \quad \text{and} \quad L - L_{cd}$$

have the same characteristic polynomial.

Similarly to vertex cospectrality, the notion of Laplacian-cospectrality can be stated in terms of the adjugate of the characteristic polynomial of L. To do so, we will need the following standard result, which we state without proof (see, for instance, [46] or [13]).

Lemma 2.3.2. For a general square matrix M,

$$\det(M + uv^T) = \det(M) + v^T \operatorname{adj}(M) u,$$

where adj denotes the adjugate of M. ■

Lemma 2.3.3. Let $ab, cd \in E$. Then ab and cd are Laplacian-cospectral if and only if

$$z_{ab}^T \operatorname{adj}(xI - L) z_{ab} = z_{cd}^T \operatorname{adj}(xI - L) z_{cd}$$

as polynomials in x.

Proof. Take $M := xI - L$ and $u := v := z_{ab}$. Then

$$\det(xI - (L - L_{ab})) = \det(xI - L + z_{ab} z_{ab}^T),$$

and, from Lemma 2.3.2, we have

$$\det(xI - (L - L_{ab})) = \det(xI - L) + z_{ab}^T \operatorname{adj}(xI - L) z_{ab}.$$

The same formula holds for the edge cd. Since the first term $\det(xI - L)$ is independent of the edge, the characteristic polynomials of $L - L_{ab}$ and $L - L_{cd}$ are equal if and only if

$$z_{ab}^T \operatorname{adj}(xI - L) z_{ab} = z_{cd}^T \operatorname{adj}(xI - L) z_{cd}.$$ ■

With this, we can prove an equivalence between edge-rigidity and Laplacian-cospectrality:

Theorem 2.3.4. All edges of G are pairwise Laplacian-cospectral if and only if G is edge-rigid.

Proof. Let $L := \sum_{i=1}^{r} \lambda_i E_i$ be the spectral decomposition of L, with $0 = \lambda_1 < \ldots < \lambda_r$, and let $m_i := \operatorname{tr}(E_i)$ be the multiplicity of λ_i. The characteristic polynomial of L is

$$p(x) := \det(xI - L) = \prod_{i=1}^{r} (x - \lambda_i)^{m_i}.$$

In the λ_i-eigenspace, the matrix $\operatorname{adj}(xI - L)$ acts as multiplication by

$$\frac{p(x)}{x - \lambda_i} = (x - \lambda_i)^{m_i - 1} \prod_{j \neq i} (x - \lambda_j)^{m_j},$$

therefore

$$\operatorname{adj}(xI - L) = \sum_{i=1}^{r} \frac{p(x)}{x - \lambda_i} E_i.$$

From Lemma 2.3.3, all edges are Laplacian-cospectral if and only if $z_{ab}^T \operatorname{adj}(xI - L) z_{ab}$ is independent of the edge ab. Using the spectral decomposition of $\operatorname{adj}(xI - L)$, we have

$$z_{ab}^T \operatorname{adj}(xI - L) z_{ab} = \sum_{i=1}^{r} \frac{p(x)}{x - \lambda_i} z_{ab}^T E_i z_{ab}.$$

We now claim that the polynomials $\{p(x)/(x - \lambda_i)\}_{i=1}^r$ are linearly independent. Indeed, the polynomial $p(x)/(x - \lambda_i)$ vanishes at $x = \lambda_i$ with order $m_i - 1$, whereas $p(x)/(x - \lambda_j)$ vanishes at $x = \lambda_i$ with order m_i for $j \neq i$. If a linear combination of said polynomials equals zero, then taking the derivative of order $m_i - 1$ at $x = \lambda_i$ isolates the i-th coefficient, which in turn implies that each coefficient has to be zero. It then follows that the polynomial $z_{ab}^T \operatorname{adj}(xI - L) z_{ab}$ is independent of ab if and only if each coefficient $z_{ab}^T E_i z_{ab}$ is independent of ab. This is precisely the statement that $\mathcal{L}^*(E_i)$ is constant for every i, which concludes the proof. ∎

The previous characterization allows us to prove edge-rigidity for edge-transitive graphs:

Corollary 2.3.5. All edge-transitive graphs are edge-rigid.

Proof. Let $ab, cd \in E$. Since G is edge-transitive, there is an automorphism σ of G mapping the edge ab to the edge cd. Let P be the permutation matrix corresponding to σ. Then $P^T L P = L$, and

$$P^T z_{ab} = \pm z_{cd},$$

hence

$$P^T L_{ab} P = P^T z_{ab} z_{ab}^T P = z_{cd} z_{cd}^T = L_{cd}.$$

Therefore $P^T(L - L_{ab})P = L - L_{cd}$, so $L - L_{ab}$ and $L - L_{cd}$ are similar, and hence cospectral. ∎

2.4 SDPs and k-conformal rigidity

In this section, we will show how to model $\max_w s_k(w)$ and $\min_w S_k(w)$ as semidefinite programs, and use the structure of their optimal solutions to study k-conformal rigidity.

We start with the well-known formulations to compute sums of eigenvalues as semidefinite programs. These are straightforward consequences of the work of Ky Fan [27], and are now standard in the semidefinite programming literature (see, for instance, [9, 15]). Define the following sets of positive semidefinite matrices:

$$\mathcal{X}_k := \{X \in \mathbb{S}^V : 0 \preceq X \preceq I,\ \operatorname{tr}(X) = k\}, \tag{2.1}$$

$$\mathcal{Z}_k := \{Z \in \mathcal{X}_k : Z\mathbb{1} = 0\}. \tag{2.2}$$

It can be shown that

$$S_k(w) = \max_{X \in \mathcal{X}_k} \operatorname{tr}(\mathcal{L}(w)X), \tag{2.3}$$

and, as $\mathcal{L}(w)$ has a zero eigenvalue with eigenvector $\mathbb{1}$, it can also be shown that

$$s_k(w) = \min_{Z \in \mathcal{Z}_k} \operatorname{tr}(\mathcal{L}(w)Z). \tag{2.4}$$

If $\mathcal{L}(w) = \sum_{i=1}^n \lambda_i v_i v_i^T$ is a spectral decomposition of $\mathcal{L}(w)$, with $\|v_i\| = 1$, then an optimal solution to (2.3) is given by

$$X = \sum_{i=n-k+1}^{n} v_i v_i^T$$

and an optimal solution to (2.4) is given by

$$Z = \sum_{i=2}^{k+1} v_i v_i^T.$$

If L has r distinct eigenvalues, and $k = k_j := \sum_{i=r-j+1}^{r} \operatorname{tr}(E_i)$ for some $j \in [r]$, then $X_j := \sum_{i=n-k+1}^{n} v_i v_i^T$ is the unique optimizer to (2.3). If $j \in [r-1]$ and $k = k_j + t$ with $0 < t < \operatorname{tr}(E_{r-j})$, then any matrix of the form $X := X_j + \tilde{E}$, where $E_{r-j} \succeq \tilde{E} \succeq 0$ and $\operatorname{tr}(\tilde{E}) = t$, is an optimal solution, and analogous results hold for the k smallest non-trivial eigenvalues. In fact, we continue our treatment below focusing on S_k, but all results have analogous versions for s_k.

Consider the problem of minimizing $S_k(w)$ over all valid edge weights:

$$\min_{w \in \Delta_E} S_k(w) = \min_{w \in \Delta_E} \max_{X \in \mathcal{X}_k} \operatorname{tr}(\mathcal{L}(w)X).$$

By writing the semidefinite dual of (2.3), we can express this as

$$\min_{w \in \Delta_E} S_k(w) = \min\{ky + \operatorname{tr}(Y) : yI + Y - \mathcal{L}(w) \succeq 0,\ Y \succeq 0,\ y \in \mathbb{R},\ w \in \Delta_E\}. \tag{2.5}$$

The corresponding dual is

$$\max\{|E|x : \mathcal{L}^*(X) \geq x\mathbb{1},\ X \in \mathcal{X}_k,\ x \in \mathbb{R}\}. \tag{2.6}$$

Note that setting $(Y = I, y = \lambda_n, w = \mathbb{1})$ and $(X = (k/n)I, x = 0)$ yields Slater points for both primal and dual programs, hence strong duality holds (see Section 1.3). By explicitly writing the complementary slackness conditions, we obtain:

Lemma 2.4.1. If (Y, y, w) and (X, x) are feasible solutions for (2.5) and (2.6), respectively, then they are optimal if and only if

$$X(Y + yI - \mathcal{L}(w)) = 0, \quad XY = Y, \quad \langle \mathcal{L}^*(X) - x\mathbb{1}, w\rangle = 0.$$

Proof. We consider the following chain of inequalities:

$$\begin{aligned}
|E|x &= \langle x\mathbb{1}, w\rangle \\
&\leq \langle \mathcal{L}^*(X), w\rangle \\
&= \langle X, \mathcal{L}(w)\rangle \\
&\leq \langle X, Y + yI\rangle \\
&= y\,\mathrm{tr}(X) + \langle X, Y\rangle \\
&\leq ky + \langle I, Y\rangle.
\end{aligned}$$

As strong duality holds, at optimal solutions equality must hold throughout the chain. Hence

$$\langle \mathcal{L}^*(X) - x\mathbb{1}, w\rangle = 0, \quad \langle X, Y + yI - \mathcal{L}(w)\rangle = 0, \quad \langle I - X, Y\rangle = 0.$$

For the latter two equalities, since both matrices in each inner product are positive semidefinite, their product must be zero (see Proposition 1.1.1), which yields the desired conditions. The converse is immediate. ■

These conditions allow us to prove the following:

Lemma 2.4.2. Suppose G is a graph with unweighted Laplacian $L = \sum_{i=1}^r \lambda_i E_i$, with $0 = \lambda_1 < \ldots < \lambda_r$. Fix $j \in [r-1]$, and let $k_j = \sum_{i=r-j+1}^r \mathrm{tr}(E_i)$ and $X_j = \sum_{i=r-j+1}^r E_i$. Then the following hold:

(1) If G is upper k_j-conformally rigid, then $\mathcal{L}^*(X_j) = \beta\mathbb{1}$ for some $\beta > 0$.

(2) If $j \in [r-2]$ and G is both upper k_j-conformally rigid and upper k_{j+1}-conformally rigid, then the canonical embedding onto the $(r-j)$-th eigenspace of L is edge-isometric. Moreover, G is also upper k-conformally rigid for all intermediate $k_j < k < k_{j+1}$.

Analogous versions of these results hold for lower conformal rigidity.

Proof. For (1), assume that G is upper k_j-conformally rigid, so $w = \mathbb{1}$ attains the minimum of $S_{k_j}(w)$ over Δ_E. Combining this with the definition of (2.6), we get that if (X, x) is optimal, then

$$S_{k_j}(\mathbb{1}) = \min_{w \in \Delta_E} S_{k_j}(w) = |E|x = \langle x\mathbb{1}, \mathbb{1}\rangle,$$

and as Lemma 2.4.1 gives $\mathcal{L}^*(X) = x\mathbb{1}$, this implies $S_{k_j}(\mathbb{1}) = \langle L, X\rangle$. From the observations at the beginning of this section, the unique optimal matrix attaining $S_{k_j}(\mathbb{1})$ in (2.3) is exactly $X_j = \sum_{i=r-j+1}^{r} E_i$, thus $X = X_j$, and setting $\beta := x$ gives $\mathcal{L}^*(X_j) = \beta\mathbb{1}$, as desired.

For (2), assume G is upper k_j-conformally rigid and upper k_{j+1}-conformally rigid. The complementary slackness conditions of Lemma 2.4.1 combined with the fact that $w = \mathbb{1} > 0$ guarantees the existence of dual solutions (X_j, x_j) and (X_{j+1}, x_{j+1}) for (2.6) such that:

$$\mathcal{L}^*(X_j) = x_j\mathbb{1}, \quad \text{with } |E|x_j = S_{k_j}(\mathbb{1}),$$
$$\mathcal{L}^*(X_{j+1}) = x_{j+1}\mathbb{1}, \quad \text{with } |E|x_{j+1} = S_{k_{j+1}}(\mathbb{1}).$$

We note that $x_j = \langle L, X_j\rangle/|E|$, and since $E_{r-j} = X_{j+1} - X_j$, setting $\gamma := x_{j+1} - x_j > 0$ gives $\mathcal{L}^*(E_{r-j}) = \gamma\mathbb{1}$, which proves the first claim. Now let k be an integer with $k_j < k < k_{j+1}$. We can write $k = k_j + \alpha\,\mathrm{tr}(E_{r-j})$, where $\alpha := (k - k_j)/\,\mathrm{tr}(E_{r-j}) \in (0, 1)$, and define

$$X_\alpha := \sum_{i=r-j+1}^{r} E_i + \alpha E_{r-j} \quad \text{and} \quad x_\alpha := \frac{\langle L, X_\alpha\rangle}{|E|}.$$

One can check that (X_α, x_α) is feasible for (2.6). Since this dual program is a maximization problem, we have for any $w \in \Delta_E$:

$$S_k(w) \geq |E|x_\alpha = S_{k_j}(\mathbb{1}) + \alpha\,\mathrm{tr}(E_{r-j})\lambda_{r-j} = S_k(\mathbb{1}),$$

hence $S_k(w) \geq S_k(\mathbb{1})$ for all $w \in \Delta_E$, proving that G is upper k-conformally rigid. ■

Before proving the main result of this section, we remark that it is immediate to notice that a graph is upper k-conformally rigid if and only if it is lower $(n-1-k)$-conformally rigid, for $k \in [n-2]$. Thus, there is no need to distinguish between upper and lower total conformal rigidity:

Proposition 2.4.3. A graph G is upper k-conformally rigid for all $k \in [n-1]$ if and only if it is lower k-conformally rigid for all $k \in [n-1]$. ■

With these auxiliary results, we can prove the following:

Theorem 2.4.4. A graph G is totally conformally rigid if and only if it is edge-rigid.

Proof. Let $0 = \lambda_1 < \ldots < \lambda_r$ be the distinct eigenvalues of L, fix $j \in [r-1]$, and let $k_j := \sum_{i=r-j+1}^{r} \mathrm{tr}(E_i)$ and $X_j := \sum_{i=r-j+1}^{r} E_i$. For the forward direction, we apply Lemma 2.4.2 for each k_j with $j \in [r-1]$ to conclude edge-rigidity.

For the backward direction, assume $\mathcal{L}^*(E_i) = \gamma_i \mathbb{1}$ with $\gamma_i > 0$. Define

$$x_j := \sum_{i=r-j+1}^{r} \gamma_i, \quad Y_j := \sum_{i=r-j+1}^{r} (\lambda_i - \lambda_{r-j}) E_i, \quad y_j := \lambda_{r-j}.$$

One can easily verify that (X_j, x_j) and $(Y_j, y_j, \mathbb{1})$ satisfy the conditions of Lemma 2.4.1, and thus form a pair of primal-dual optimal solutions. Since their objective value is $S_{k_j}(\mathbb{1})$, this implies that G is upper k_j-conformally rigid for all $j \in [r-1]$. Combining this with Lemma 2.4.2 allows us to conclude that G is upper k-conformally rigid for all $k \in [n-1]$, which by the previous proposition implies that G is totally conformally rigid. ■

Combining Theorems 2.3.4 and 2.4.4 gives us a four-way equivalence:

Corollary 2.4.5. For a graph G, the following are equivalent:

(a) All edges of G are pairwise Laplacian-cospectral.

(b) G is edge-rigid.

(c) G is totally conformally rigid.

(d) For every Laplacian eigenprojector E_i, the vector $\mathcal{L}^*(E_i)$ is constant.

■

2.5 Laplacian powers and line graphs

In this section, we give a combinatorial characterization of total conformal rigidity in terms of powers of the Laplacian matrix.

Theorem 2.5.1. A graph G is edge-rigid if and only if for every integer $\ell \geq 0$, there exists a constant $C_\ell \in \mathbb{R}$ such that

$$\mathcal{L}^*(L^\ell)_{ab} = L^\ell_{aa} + L^\ell_{bb} - 2L^\ell_{ab} = C_\ell \quad \text{for every } ab \in E.$$

Proof. For the forward direction, assume G is edge-rigid. The orthogonal projectors onto the eigenspaces of L satisfy $\mathcal{L}^*(E_i) = \gamma_i \mathbb{1}$ for some $\gamma_i > 0$ and each $i \in \{2, \ldots, r\}$. Using the spectral decomposition $L = \sum_{i=1}^{r} \lambda_i E_i$, we can write

$$L^\ell = \sum_{i=1}^{r} \lambda_i^\ell E_i.$$

Applying the linear operator $\mathcal{L}^*$ yields

$$\mathcal{L}^*(L^\ell) = \sum_{i=1}^{r} \lambda_i^\ell \mathcal{L}^*(E_i) = \left(\sum_{i=1}^{r} \lambda_i^\ell \gamma_i\right) \mathbb{1}.$$

Noting that $\lambda_1 = 0$ and setting $C_\ell := \sum_{i=2}^{r} \lambda_i^\ell \gamma_i$ gives the desired implication.

For the reverse direction, assume $\mathcal{L}^*(L^\ell) = C_\ell \mathbb{1}$ for every integer $\ell \geq 0$. By applying $\mathcal{L}^*$ to the spectral decomposition, we obtain a linear system for $\ell \in \{0, 1, \ldots, r-1\}$:

$$\sum_{i=1}^{r} \lambda_i^\ell \mathcal{L}^*(E_i) = C_\ell \mathbb{1}.$$

Because the r eigenvalues $\lambda_1, \ldots, \lambda_r$ are distinct, the coefficient matrix of this system is an $r \times r$ Vandermonde matrix, which is invertible. Consequently, each vector $\mathcal{L}^*(E_i) \in \mathbb{R}_+^E$ can be uniquely expressed as a linear combination of the constant vectors $\{C_0\mathbb{1}, C_1\mathbb{1}, \ldots, C_{r-1}\mathbb{1}\}$. This implies that $\mathcal{L}^*(E_i) = \gamma_i \mathbb{1}$ for some $\gamma_i \in \mathbb{R}$ and every $i \in \{2, \ldots, r\}$, hence G is edge-rigid. ■

By noting that a walk-regular graph must also have L^ℓ with constant diagonals for all $\ell \geq 0$, and also that such Laplacian powers must also be constant on the entries corresponding to edges for 1-walk-regular graphs, we can conclude the following:

Corollary 2.5.2. If G is walk-regular, then it is edge-rigid if and only if it is 1-walk-regular. ■

Example 2.5.3. There are edge-rigid graphs that are not edge-transitive. Indeed, every strongly regular graph is 1-walk-regular, and by the Corollary above every strongly regular graph is edge-rigid. Consequently, any strongly regular graph that is not edge-transitive gives an example. For instance, the Chang graphs [20] on 28 vertices are strongly regular with parameters $(28, 12, 6, 4)$ and are not edge-transitive.

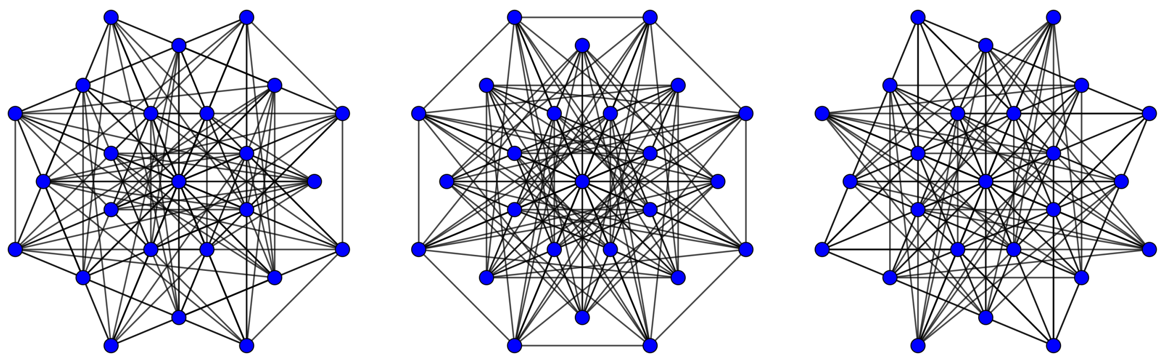

Fig. 2.1: The three Chang graphs [20], which are edge-rigid but not edge-transitive.

In [70], the authors utilize semidefinite programming to numerically test and certify wether a graph is conformally rigid. While semidefinite programs can be solved to within any prescribed tolerance in polynomial time [60], this continuous optimization approach only provides an approximate numerical verification rather than an exact combinatorial decision procedure, and in practice it is susceptible to numerical instability and precision artifacts on larger instances.

An immediate and significant consequence of Theorem 2.5.1 is that total conformal rigidity can be decided exactly in deterministic polynomial time via a straightforward algebraic algorithm that requires only integer arithmetic. This is achieved by computing $\mathcal{L}^*(L^\ell)$ for all $\ell \in [n-1]$ and checking whether each resulting vector is constant. Because the eigenvalues of the unweighted Laplacian L are bounded above by $2n$, the entries of L^ℓ for $\ell \in [n-1]$ are integers bounded in absolute value by $(2n)^{n-1}$. Consequently, the arithmetic computations involve integers whose bit-length grows at most linearly with $n \log n$. Since computing the sequence of matrix powers $L, L^2, \ldots, L^{n-1}$ requires $O(n^4)$ elementary arithmetic operations, the entire procedure can be implemented exactly without numerical error in polynomial time with respect to the bit-complexity model.

Corollary 2.5.4. If $G = (V, E)$ is a graph on n vertices, then there is a deterministic polynomial-time algorithm for deciding if G is edge-rigid using $O(n^4)$ arithmetic operations over integers of bit-length $O(n \log n)$. ■

The structural constraints imposed by edge-rigidity can be translated into the language of line graphs. To make this precise, fix an arbitrary orientation of the edges of G, and construct the oriented incidence matrix $B \in \mathbb{R}^{V \times E}$ such that the column corresponding to an edge $e = \{u, v\}$ directed from v to u is $z_e = e_u - e_v$. While the Laplacian $L = BB^T$ is independent of this orientation, the product $B^T B \in \mathbb{R}^{E \times E}$ depends on the choice of orientation. The diagonal entries of $B^T B$ are $z_e^T z_e = 2$. For distinct edges e and f, the entry $(B^T B)_{ef}$ is ± 1 if they share a vertex, and 0 otherwise. Thus, we may write $B^T B = 2I + A_\sigma$, where A_σ is the adjacency matrix of a signed line graph.

The exact signature σ depends on the chosen orientation of G. However, reversing the orientation of an edge $e \in E$ simply negates the column z_e in B. This operation conjugates $B^T B$ by a diagonal matrix D with $D_{ee} = -1$ and $D_{ff} = 1$ for all $f \neq e$. In the context of signed graphs, this corresponds exactly to a vertex switching in the signed line graph. Since switching-equivalent signed graphs have similar adjacency matrices, the diagonal entries of $(A_\sigma^\ell)_{ee}$ are invariant under the choice of orientation for every $\ell \geq 0$. We can thus prove the following:

Corollary 2.5.5. A graph G is edge-rigid if and only if, for any orientation of its edges, the corresponding signed line graph of G is walk-regular.

Proof. By Theorem 2.5.1, G is edge-rigid if and only if for every integer $\ell \geq 0$, $\mathcal{L}^*(L^\ell)_{ab} = C_\ell$ for all $ab \in E$. Using the factorization $L = BB^T$, we rewrite the adjoint operator acting on L^ℓ for a specific edge $e = \{a, b\} \in E$. Let $e_{ab} \in \mathbb{R}^E$ denote the standard basis vector for this edge, so that the corresponding column in the oriented incidence matrix is $Be_{ab} = z_{ab}$. Then:

$$\mathcal{L}^*(L^\ell)_{ab} = z_{ab}^T(BB^T)^\ell z_{ab} = (Be_{ab})^T(BB^T)^\ell(Be_{ab}) = e_{ab}^T(B^TB)^{\ell+1}e_{ab}.$$

This final expression is exactly the ab-diagonal entry of the matrix $(B^TB)^{\ell+1}$. Therefore, $\mathcal{L}^*(L^\ell)_{ab}$ is a global constant C_ℓ for all edges $ab \in E$ and all integers $\ell \geq 0$ if and only if the diagonal of $(B^TB)^m$ is constant for all $m \geq 1$. Because $B^TB = 2I + A_\sigma$, this is equivalent to the diagonal entries of A_σ^m being constant for all $m \geq 0$. Since this invariance holds regardless of the initial orientation chosen for G, this is exactly the definition of the corresponding line graph being walk-regular. ■

2.6 Walk-regularity and biregularity

In this section, we further explore the connection between edge-rigidity and walk-regularity: we show that all regular edge-rigid graphs are walk-regular, and provide analogous results for bipartite biregular edge-rigid graphs.

We first establish a combinatorial consequence of edge-rigidity:

Lemma 2.6.1. If G is a connected edge-rigid graph, then the sum of the degrees of the endpoints of any edge is a global constant across the graph. Consequently, G must be either regular or biregular bipartite.

Proof. By Theorem 2.3.4, the orthogonal projectors E_i onto the eigenspaces of the unweighted Laplacian L satisfy $\mathcal{L}^*(E_i) = \gamma_i\mathbb{1}$ for scalars $\gamma_i > 0$. Writing $L = \sum_{i=1}^r \lambda_i E_i$, it follows that $\mathcal{L}^*(L) = C\mathbb{1}$ for some $C \in \mathbb{R}$. For any edge $ab \in E$,

$$C = \mathcal{L}^*(L)_{ab} = L_{aa} + L_{bb} - 2L_{ab} = d_a + d_b + 2,$$

where d_a denotes the degree of vertex a, so $d_a + d_b$ is a global constant across all edges.

Now consider any two vertices a and c connected by a walk of length two through an intermediate vertex b. We have $d_a + d_b = d_b + d_c$, which implies $d_a = d_c$. If a and c are in turn connected by a walk of length 2ℓ, for some $\ell \geq 1$, we can use the previous case as the base of an induction to easily conclude that any two vertices connected by walks of even length must share the same degree. If the graph contains an odd cycle, then any two vertices are connected by a walk of even length, because the graph is connected and the odd cycle may be used to choose the parity of a walk connecting any two vertices. So all vertices have the same degree and G is regular. If the graph contains no odd cycles, it is bipartite, and the two sides of the bipartition form two degree classes, making G biregular bipartite. ■

Thus every edge-rigid graph is either regular or biregular bipartite. We handle the regular case first:

Theorem 2.6.2. If G is a regular edge-rigid graph, then G is walk-regular.

Proof. Assume G is a d-regular edge-rigid graph with adjacency matrix A. Let $W_\ell(a) = (A^\ell)_{aa}$ denote the number of closed walks of length ℓ starting at vertex $a \in V$, and let $W_\ell \in \mathbb{R}^V$ be the vector of these walk counts.

Because G is d-regular, its unweighted Laplacian is $L = dI - A$. By Theorem 2.5.1, edge-rigidity implies that for any edge $ab \in E$, $\mathcal{L}^*(L^\ell)_{ab} = C_\ell$ for some constant C_ℓ and all $\ell \geq 0$. Since A^ℓ can be expanded as a polynomial in L, linearity implies that $\mathcal{L}^*(A^\ell)$ is also a constant vector. That is, there exist constants C'_ℓ such that for any edge $ab \in E$,

$$\mathcal{L}^*(A^\ell)_{ab} = W_\ell(a) + W_\ell(b) - 2(A^\ell)_{ab} = C'_\ell.$$

Summing over all d neighbors b adjacent to a:

$$\sum_{b\sim a} W_\ell(a) + \sum_{b\sim a} W_\ell(b) - 2\sum_{b\sim a}(A^\ell)_{ab} = dC'_\ell.$$

The first term equals $dW_\ell(a)$ and the second equals $(AW_\ell)_a$. For the third term, we observe that

$$\sum_{b\sim a}(A^\ell)_{ab} = \sum_{b\in V} A_{ab}(A^\ell)_{ba} = (A^{\ell+1})_{aa} = W_{\ell+1}(a).$$

Combining these, we obtain the recurrence

$$dW_\ell + AW_\ell - 2W_{\ell+1} = dC'_\ell \mathbb{1}.$$

We now proceed by induction on ℓ to show that W_ℓ is a constant vector for all $\ell \geq 0$. The base case $\ell = 0$ is trivial since $W_0 = \mathbb{1}$. Assume $W_\ell = \alpha_\ell \mathbb{1}$ for some scalar α_ℓ. Because G is d-regular, $AW_\ell = A(\alpha_\ell \mathbb{1}) = d\alpha_\ell \mathbb{1}$. Substituting into the recurrence yields

$$2W_{\ell+1} = d(2\alpha_\ell - C'_\ell)\mathbb{1},$$

so $W_{\ell+1}$ is also a scalar multiple of $\mathbb{1}$. By induction, the number of closed walks of any length is constant across all vertices, hence G is walk-regular. ■

Combining the previous result with Corollary 2.5.2 gives us a combinatorial characterization of regular edge-rigid graphs:

Corollary 2.6.3. If G is a regular graph, then it is edge-rigid if and only if it is 1-walk-regular. ■

We now turn to edge-rigid graphs that are biregular bipartite. First, we introduce an analogous notion of walk-regularity for this case:

Definition 2.6.4. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. We say G is *walk-biregular* if, for every integer $\ell \geq 0$, there exist constants $\alpha_\ell^{(1)}$ and $\alpha_\ell^{(2)}$ such that the number of closed walks of length ℓ starting at any vertex $a \in V_1$ is exactly $\alpha_\ell^{(1)}$, and starting at any vertex $b \in V_2$ is exactly $\alpha_\ell^{(2)}$. Furthermore, we say G is 1-*walk-biregular* if it is walk-biregular and the number of walks of length ℓ between the endpoints of any edge $ab \in E$ is a global constant β_ℓ depending only on ℓ.

Note that because G is bipartite, $\alpha_\ell^{(1)} = \alpha_\ell^{(2)} = 0$ for all odd ℓ, and $\beta_\ell = 0$ for all even ℓ. We can now state and prove the bipartite counterpart to the regularity result above.

Theorem 2.6.5. If $G = (V_1 \cup V_2, E)$ is a biregular bipartite edge-rigid graph, then G is walk-biregular.

Proof. Assume G is connected and biregular bipartite with vertex partitions V_1 and V_2, where vertices in V_1 have degree d_1 and vertices in V_2 have degree d_2. We express the degree matrix as $D = d_1 I_1 + d_2 I_2$, where I_1 and I_2 are the diagonal indicator matrices for V_1 and V_2, respectively.

Let $U_\ell(a) := (L^\ell)_{aa}$ and $U_\ell \in \mathbb{R}^V$. We will show by induction on ℓ that U_ℓ is constant on V_1 and on V_2 for every $\ell \geq 0$. The base case of $\ell = 0$ is immediate, since $L^0 = I$. Assume the claim holds for all $k \leq \ell$. By Theorem 2.5.1, $\mathcal{L}^*(L^\ell)_{ab} = C_\ell$ for every edge ab with $a \in V_1$ and $b \in V_2$:

$$U_\ell(a) + U_\ell(b) - 2(L^\ell)_{ab} = C_\ell. \tag{2.7}$$

By the inductive hypothesis, $U_\ell(a) = \gamma_\ell^{(1)}$ for all $a \in V_1$ and $U_\ell(b) = \gamma_\ell^{(2)}$ for all $b \in V_2$, so Equation (2.7) shows that $(L^\ell)_{ab}$ is a global constant η_ℓ for all edges $ab \in E$.

Now consider the diagonal of $L^{\ell+1} = LL^\ell = (D - A)L^\ell$. For any $a \in V_1$:

$$U_{\ell+1}(a) = d_1 (L^\ell)_{aa} - \sum_{b \sim a} (L^\ell)_{ab} = d_1 \gamma_\ell^{(1)} - d_1 \eta_\ell = d_1(\gamma_\ell^{(1)} - \eta_\ell),$$

which is constant for all $a \in V_1$. A symmetric argument gives $U_{\ell+1}(b) = d_2(\gamma_\ell^{(2)} - \eta_\ell)$, constant for all $b \in V_2$. By induction, the diagonal of L^ℓ is constant on each part for all $\ell \geq 0$.

We must now show that this implies G is walk-biregular. As the entries of A^ℓ count the number of walks of length ℓ between vertices, it follows that for any $a \in V_1$, $A^\ell e_a$ is entirely supported on V_1 for even ℓ, and on V_2 for odd ℓ. Consequently, $DA^\ell e_a = d_1 A^\ell e_a$ for even ℓ, and $DA^\ell e_a = d_2 A^\ell e_a$ for odd ℓ. Because $L = D - A$, the vector $L^\ell e_a = (D - A)^\ell e_a$ can be expanded as a polynomial in D and A. By evaluating from right to left, every instance of D acts as either d_1 or d_2 depending on the parity of the number of A's that have already been applied. Therefore, $L^\ell e_a$ simplifies to a linear combination $\sum_{i=0}^{\ell} c_{i,\ell}^{(1)} A^i e_a$, where the coefficients depend only on d_1, d_2, and ℓ. By a symmetric argument, $(D-L)^\ell e_a$ also evaluates to a linear combination of $L^i e_a$. Thus, $(A^\ell)_{aa} = e_a^T A^\ell e_a$ is a linear combination of the entries $(L^i)_{aa}$ for $i \leq \ell$. Since we established by induction that $(L^i)_{aa}$ is constant across V_1 (and similarly for V_2), it follows that the diagonal entries of A^ℓ are constant on V_1 and V_2, proving that G is walk-biregular. ■

Similarly to the regular case, we can prove the following:

Proposition 2.6.6. Let G be a walk-biregular graph. Then G is edge-rigid if and only if it is 1-walk-biregular.

Proof. Suppose G is 1-walk-biregular. Then for any $\ell \geq 0$, the diagonal entries of A^ℓ are constant on V_1 and V_2, and the off-diagonal entries $(A^\ell)_{ab}$ are a global constant across all edges $ab \in E$. Since $L^\ell e_a = (D-A)^\ell e_a$ can be expressed as $\sum_{i=0}^{\ell} c_{i,\ell}^{(1)} A^i e_a$ (and similarly for e_b), the entry $(L^\ell)_{ab} = e_a^T L^\ell e_b$ is a linear combination of the edge entries $(A^i)_{ab}$. Because the terms $(A^i)_{ab}$ are constant across all edges, $(L^\ell)_{ab}$ is also constant for all $ab \in E$. The same applies to the diagonals $(L^\ell)_{aa}$ and $(L^\ell)_{bb}$. Consequently, $\mathcal{L}^*(L^\ell)_{ab} = (L^\ell)_{aa} + (L^\ell)_{bb} - 2(L^\ell)_{ab}$ evaluates to a global constant C_ℓ for all $\ell \geq 0$. By Theorem 2.5.1, G is edge-rigid.

Now suppose G is edge-rigid. By Theorem 2.5.1, $\mathcal{L}^*(L^\ell)_{ab}$ is a global constant C_ℓ for all $\ell \geq 0$. Since G is walk-biregular, we know that the diagonal entries of A^ℓ are constant on V_1 and V_2, which by an analogous argument to the one above implies that the diagonal entries of L^ℓ must also be constant on V_1 and V_2. Since $\mathcal{L}^*(L^\ell)_{ab}$ and the diagonals are constant, $(L^\ell)_{ab}$ must be constant across all edges $ab \in E$. Finally, expanding $A^\ell e_a = (D-L)^\ell e_a$ as a linear combination of $L^i e_a$ demonstrates that $(A^\ell)_{ab}$ is a linear combination of the constant edge entries $(L^i)_{ab}$. Therefore, $(A^\ell)_{ab}$ is a global constant across all edges, meaning G is 1-walk-biregular. ■

Combining the two cases yields a complete combinatorial characterization of edge-rigidity:

Corollary 2.6.7. A graph G is edge-rigid if and only if it is either 1-walk-regular or 1-walk-biregular. ■

2.7 Majorization and combinatorial consequences

In this short section, we show two non-trivial combinatorial consequences of total conformal rigidity, obtained upon applying the theory of majorization of vectors (see Bhatia [13, Chapter II] for a standard reference). Recall that given two vectors $x, y \in \mathbb{R}^n$ ordered such that $x_1 \geq \ldots \geq x_n$ and $y_1 \geq \ldots \geq y_n$, we say that x is majorized by y if

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad \text{for all } k \in [n] \quad \text{and} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

Let $\operatorname{supp}(w) := \{e \in E : w_e > 0\}$ denote the support of a weight assignment $w \in \Delta_E$. Because edge weights are permitted to be zero, a choice of weights may result in a weighted graph that is disconnected even though the underlying simple graph G is connected. We thus define the set of non-disconnecting valid edge weights as

$$\Delta_E^+ := \{w \in \Delta_E : (V, \operatorname{supp}(w)) \text{ is connected}\},$$

which includes all strictly positive weight assignments. Equivalently, Δ_E^+ consists of all weight vectors whose support contains the edge set of at least one spanning tree of G. This combinatorial characterization is precisely what guarantees that the algebraic connectivity of the weighted graph remains strictly positive. That is, for any $w \in \Delta_E$, we have $w \in \Delta_E^+$ if, and only if,

$$0 = \lambda_1(w) < \lambda_2(w) \leq \ldots \leq \lambda_n(w),$$

meaning the weighted Laplacian $\mathcal{L}(w)$ maintains rank $n-1$. The results derived in this section rely on the following key observation: G being totally conformally rigid is precisely equivalent to the vector of eigenvalues of L being majorized by the vector of eigenvalues of $\mathcal{L}(w)$ for any $w \in \Delta_E^+$.

The weighted matrix tree theorem [48] states that for a given graph G with edge weights $w \in \mathbb{R}^E$,

$$\tau(G, w) := \sum_{T} \prod_{e \in T} w_e = \frac{1}{n} \prod_{i=2}^{n} \lambda_i(w),$$

where the sum is over all spanning trees T of G. We use this to prove the following:

Theorem 2.7.1. If G is totally conformally rigid, then $w = \mathbb{1}$ maximizes $\tau(G, w)$ over all $w \in \Delta_E$.

Proof. As G is totally conformally rigid, it follows that for any $w \in \Delta_E^+$, the eigenvalues of L are majorized by the eigenvalues of $\mathcal{L}(w)$. Since log is a strictly concave function on $\mathbb{R}_{++}$, we can apply [13, Theorem II.3.1] to conclude

$$\sum_{i=2}^{n} \log \lambda_i(\mathbb{1}) \geq \sum_{i=2}^{n} \log \lambda_i(w).$$

Taking exponentials on both sides and applying the matrix tree theorem yields $\tau(G, \mathbb{1}) \geq \tau(G, w)$ for all $w \in \Delta_E^+$. For weights $w \in \Delta_E \setminus \Delta_E^+$, the graph is disconnected, yielding $\tau(G, w) = 0 \leq \tau(G, \mathbb{1})$. Thus, the maximum over the entire simplex is achieved at $w = \mathbb{1}$. ∎

The Kirchhoff index of a weighted graph G is defined as

$$\mathrm{Kf}(G, w) := \sum_{\{a,b\} \subseteq V} z_{ab}^T \mathcal{L}(w)^\dagger z_{ab},$$

where $\mathcal{L}(w)^\dagger$ is the Moore-Penrose pseudoinverse of the weighted Laplacian $\mathcal{L}(w)$ and $z_{ab} = e_a - e_b$. The term $z_{ab}^T \mathcal{L}(w)^\dagger z_{ab}$ is the effective resistance between vertices a and b (see [49] for a standard reference). If the graph is disconnected, we conventionally define $\mathrm{Kf}(G, w) = \infty$.

Theorem 2.7.2. If G is totally conformally rigid, then $w = \mathbb{1}$ minimizes $\mathrm{Kf}(G, w)$ over all $w \in \Delta_E$.

Proof. For $w \in \Delta_E \setminus \Delta_E^+$, the graph is disconnected and $\mathrm{Kf}(G, w) = \infty$, which is trivially suboptimal. For $w \in \Delta_E^+$, one can express the Kirchhoff index as

$$\mathrm{Kf}(G, w) = n \sum_{i=2}^{n} \frac{1}{\lambda_i(w)},$$

which follows from an elementary expansion of the definition (see [42]). As G is totally conformally rigid, the eigenvalues of L are majorized by the eigenvalues of $\mathcal{L}(w)$ for any $w \in \Delta_E^+$. Since $x \mapsto 1/x$ is a convex function on $\mathbb{R}_{++}$, again applying [13, Theorem II.3.1] yields

$$\sum_{i=2}^{n} \frac{1}{\lambda_i(\mathbb{1})} \leq \sum_{i=2}^{n} \frac{1}{\lambda_i(w)},$$

giving $\mathrm{Kf}(G, \mathbb{1}) \leq \mathrm{Kf}(G, w)$, which completes the proof. ∎

2.8 Gauge duality for sums of eigenvalues

As a final perspective on edge-rigidity, we consider the notion of gauges introduced in Section 1.4 in the context of the parameter $S_k(w)$. We recall the following minimization expression for $S_k(w)$ obtained by SDP strong duality from (2.3) in Section 2.4:

$$S_k(w) = \min\{ky + \mathrm{tr}(Y) : yI + Y - \mathcal{L}(w) \succeq 0,\ Y \succeq 0,\ y \in \mathbb{R}\}. \tag{2.8}$$

With this, we can show the following:

Lemma 2.8.1. The function $S_k : \mathbb{R}_+^E \mapsto \mathbb{R}$ is a positive definite monotone gauge.

Proof. Positive (semi)definiteness is immediate, as $\mathcal{L}(w) \succeq 0$ for all $w \in \mathbb{R}_+^E$, and $S_k(w) = 0$ implies that $\mathcal{L}(w) = 0$, which in turn implies $w = 0$. Positive homogeneity follows from the definition of S_k and the fact that scaling w by α scales $\mathcal{L}(w)$ by α. To check subadditivity, let $w, w' \in \mathbb{R}_+^E$ and let (Y, y) and (Y', y') be the optimal solutions of the minimization SDPs for $S_k(w)$ and $S_k(w')$ (defined in (2.8)), respectively. Then $(Y + Y', y + y')$ is also a feasible solution for tha same SDP for $S_k(w + w')$ with objective value $S_k(w) + S_k(w')$, so $S_k(w + w') \leq S_k(w) + S_k(w')$. Finally, monotonicity follows from the fact that if $w \leq w'$, then $\mathcal{L}(w') - \mathcal{L}(w) = \mathcal{L}(w' - w) \succeq 0$, so any feasible solution for the maximization SDP of $S_k(w)$ (defined in (2.3)) is also feasible for the SDP of $S_k(w')$, with an objective value at least that of the latter. ■

Recall from (2.1) the definition $\mathcal{X}_k = \{X \in \mathbb{S}^V : 0 \preceq X \preceq I,\ \operatorname{tr}(X) = k\}$, and consider its image under $\mathcal{L}^*$:

$$\mathcal{L}^*(\mathcal{X}_k) = \{\mathcal{L}^*(X) : X \in \mathcal{X}_k\}.$$

The unit convex corner associated to S_k° is the downward hull of $\mathcal{L}^*(\mathcal{X}_k)$, and the unit convex corner associated to S_k is also semidefinite representable.

Proposition 2.8.2. Consider the gauge S_k, and let S_k° be its dual gauge. Let B_k be the unit convex corner associated to S_k, and let B_k° be the convex corner associated to S_k°. Then these sets are given by:

$$B_k^\circ = \left\{w \in \mathbb{R}_+^E : \exists\, y \in \mathcal{L}^*(\mathcal{X}_k) \text{ such that } w \leq y\right\}, \tag{2.9}$$

$$B_k = \{w \geq 0 : \exists\, t \in \mathbb{R},\ Z \succeq 0 \text{ such that } tI + Z \succeq \mathcal{L}(w),\ kt + \operatorname{tr}(Z) \leq 1\}. \tag{2.10}$$

Proof. We briefly comment on this proof as it is a straightforward application of the definitions and standard duality theory. The first equality follows from monotonicity and the fact that

$$S_k(w) = \max_{X \in \mathcal{X}_k} \langle \mathcal{L}(w), X\rangle = \max_{X \in \mathcal{X}_k} w^T \mathcal{L}^*(X) = \max_{y \in \mathcal{L}^*(\mathcal{X}_k)} w^T y.$$

The second equality follows from the minimization SDP formulation and the Minkowski functional presentation of the gauge S_k. ■

The result below follows from a standard argument of gauge minimization in a simplex, but we include it here with its elementary proof in order to highlight the connection between the gauges S_k and S_k° and the notion of upper k-conformal rigidity.

Theorem 2.8.3. A graph G is upper k-conformally rigid if and only if

$$S_k(\mathbb{1}) S_k^\circ(\mathbb{1}) = |E|.$$

Proof. Recall that $\Delta_E := \{w \in \mathbb{R}^E : w \geq 0,\ \mathbb{1}^T w = |E|\}$. We start this proof with the following chain of equalities:

$$\min_{w\in\Delta_E} S_k(w) = \min_{w\in\Delta_E} \max_{y\in B_k^\circ} w^T y \tag{2.11}$$
$$= \max_{y\in B_k^\circ} \min_{w\in\Delta_E} w^T y \tag{2.12}$$
$$= \max_{y\in B_k^\circ} \left\{ |E| \min_{e\in E} y_e \right\} \tag{2.13}$$
$$= |E| \max_{\lambda\geq 0} \{\lambda : \lambda\mathbb{1} \in B_k^\circ\} \tag{2.14}$$
$$= |E| \frac{1}{S_k^\circ(\mathbb{1})}. \tag{2.15}$$

The first equality follows from the fact that S_k is the dual of S_k°. The second equality follows from the minimax theorem, since the function $w^T y$ is linear in both w and y, and the sets Δ_E and B_k° are convex and compact. The third equality follows from the fact that $\min_{w\in\Delta_E} w^T y$, for a fixed y, is achieved at a vertex of Δ_E, which corresponds to putting all weight on a single edge. The fourth equality follows from the fact that B_k° is lower comprehensive, as $y \in B_k^\circ$ and $\lambda \leq \min_{e\in E} y_e$ implies $\lambda\mathbb{1} \in B_k^\circ$. Finally, the last equality follows from the definition of the dual gauge S_k°, since $S_k^\circ(\mathbb{1}) = \min_\lambda\{\lambda : \mathbb{1} \in \lambda B_k^\circ\}$.

The equivalence follows directly: G is upper k-conformally rigid if and only if $S_k(\mathbb{1}) = \min_{w\in\Delta_E} S_k(w)$, and by (2.15) this is equivalent to

$$S_k(\mathbb{1}) = \frac{|E|}{S_k^\circ(\mathbb{1})}.$$

∎

The next chapter shall study the equality cases of similar inequalities to the one discussed above in the context of MAXCUT, FCC and MAX 2-SAT, and their related gauges expressed as semidefinite programs. We shall see that not only are the techniques developed in this chapter also applicable to the study of these combinatorial parameters, but that the class of graphs that characterizes total conformal rigidity—namely 1-walk-regular and 1-walk-biregular graphs—are also sufficient for obtaining tight bounds involving some of the pairs of primal-dual gauges related to the aforementioned parameters.

Chapter 3

Bounds to FCC and MAX 2-SAT

In this chapter, we use semidefinite programs to obtain new bounds for the fractional cut-cover and maximum 2-sat parameters of graphs with high structural regularity. We use the unique structure of certain algebras associated with these graphs, coupled with properties of gauge duality, to explicitly compute optimal solutions for the semidefinite relaxations of the aforementioned problems. The contents of this chapter were originally published in [3] and are joint work with Gabriel Coutinho.

3.1 Overview

Let $G = (V, E)$ be a simple, undirected graph with adjacency matrix A. The well-known maximum cut problem (MAXCUT) consists of finding a partition of V into two sets that maximizes the number of edges between them, and we can express this problem as the following quadratic program:

$$\mathrm{mc}(G) := \max\left\{\frac{1}{4}\langle L, xx^T\rangle : x \in \mathbb{R}^V, x_i^2 = 1\right\}. \tag{3.1}$$

As in the previous chapter, we let $L = \mathcal{L}(\mathbb{1})$ be the Laplacian of G, and $\langle\cdot,\cdot\rangle$ be the trace inner product in $M_V(\mathbb{R})$. The celebrated Goemans-Williamson algorithm [38] shows how to approximate MAXCUT within a factor of $\alpha_{\mathrm{GW}} \approx 0.878$ by constructing cuts based on solutions to the SDP relaxation of (3.1) given by:

$$\eta(G) := \max\left\{\frac{1}{4}\langle L, M\rangle : M \succeq 0, \mathrm{diag}(M) = \mathbb{1}\right\}. \tag{3.2}$$

When considering the weighted versions of these parameters, they can be easily seen to satisfy the notion of gauge duality discussed in Section 1.4. In this chapter, we will be interested in studying graphs with constant edge weights, so we take the shortcut of defining the dual graph parameters the special case of the dual gauge functions taken on weights everywhere equal to 1.

The antiblocking dual parameter to MAXCUT is the fractional cut-cover number (FCC), which is defined as

$$\mathrm{fcc}(G) := \min\left\{ \mathbb{1}^T y : y \in \mathbb{R}_+^{\mathcal{P}(V)}, \sum_{S \subseteq V} y_S \cdot \mathbb{1}_{\delta(S)} \geq \mathbb{1} \right\}, \tag{3.3}$$

where $\mathcal{P}(V)$ is the power set of V, $\delta(S)$ is the set of edges with one end in S and the other in $V \setminus S$, and $\mathbb{1}_{\delta(S)}$ is the indicator vector of $\delta(S)$. The integer solutions for the previous program provide a covering of the edges of the graph by cuts, and the minimum number of such cuts is known as the cut-cover number (CC). Computing CC arises in certain tests for short circuits in printed circuit boards [53], and finding exact or even approximate solutions is known to be NP-hard [71, 59]. The FCC parameter was first introduced by Šámal [67], who employed it to study graph homomorphisms and cut-continuous functions. Neto and Ben-Ameur [62] later demonstrated how to construct a polynomial-time computable approximation for FCC by means of vector colourings, and also showed how this problem is related to frequency allocation.

Recently, the authors of [11] have shown how to construct an efficient approximation algorithm for the weighted version of FCC with factor $1/\alpha_{\mathrm{GW}} \approx 1.139$, by combining the Goemans-Williamson algorithm with the dual gauge parameter of $\eta(G)$ given by:

$$\eta^\circ(G) := \min\{\mu : \mu \geq 0, N \succeq 0, \mathrm{diag}(N) = \mu \cdot \mathbb{1}, \frac{1}{4}\mathcal{L}^*(N) \geq \mathbb{1}\}, \tag{3.4}$$

where we recall that $\mathcal{L}^*$ denotes the adjoint of the Laplacian of G given by $\mathcal{L}^*(N)_{ab} = N_{aa} + N_{bb} - 2N_{ab}$ for an edge $ab \in E$. In order to derive the previous expression, we can first consider the SDP dual program to the formulation of $\eta(G)$ in (3.2):

$$\eta(G) = \min\left\{\rho : \rho \geq 0, x \in \mathbb{R}^V, \rho \geq \mathbb{1}^T x, \mathrm{Diag}(x) \succeq \frac{1}{4}L\right\}. \tag{3.5}$$

As mentioned in Section 2.8, the dual gauge function can be defined analogously to a dual norm as

$$\eta^\circ(G) = \max\{\mathbb{1}^T w : w \in \mathbb{R}_+^E, x \in \mathbb{R}, \mathrm{Diag}(x) - \frac{1}{4}\mathcal{L}(w) \succeq 0, \mathbb{1}^T x \leq 1\},$$

where we note that the constraints are obtained by requiring a solution to the weighted version of (3.5) to have objective value at most 1. The program given in (3.4) is promptly obtained by SDP strong duality.

Using the framework of gauge duality, one can show that the pairs $\mathrm{mc}(G), \mathrm{fcc}(G)$ and $\eta(G), \eta^\circ(G)$ satisfy

$$|E| \leq \mathrm{mc}(G)\mathrm{fcc}(G), \tag{3.6a}$$

$$|E| \leq \eta(G)\eta^\circ(G), \tag{3.6b}$$

and it can also be shown that equality is attained in both cases for edge-transitive graphs (for a study of the equality cases of similar equations as the above involving the Lovász Theta, see [19]).

In this context, we are interested in studying the optimal solutions for the semidefinite programs $\eta(G), \eta^\circ(G)$ when G presents high structural regularity in order to better understand their relationship with MAXCUT and FCC. We are also interested in studying the solutions of more general semidefinite programs that can be used to approximate problems such as MAX 2-SAT.

3.2 MAXCUT and FCC

3.2.1 Graphs in coherent configurations

In this section we study the equality cases of (3.6b), which will prove to be useful for deriving the desired spectral bounds for FCC and MAX 2-SAT. Our analysis relies on one key result about the coherent algebras discussed in Section 1.2.1: the orthogonal projection onto a coherent algebra preserves positive semidefiniteness, i.e., if M is positive semidefinite, then the orthogonal projection M' onto a coherent algebra $\mathcal{A}$ is also positive semidefinite (see [7, Corollary 9.1]).

We begin with an auxiliary result that allows us to assume the existence of optimal solutions for (3.2) and (3.4) in a given coherent algebra, provided that some technical conditions are satisfied. The existence of such solutions is known when the underlying coherent algebra contains the matrices that define the SDP (see for instance [37, 50, 7, 24]), however in our case this is not true in general and an adaptation is required, as we show below.

Lemma 3.2.1. Let G be a graph with Laplacian matrix L, and let $\mathcal{C} = \{A_0, ..., A_d\}$ be a coherent configuration with coherent algebra $\mathcal{A}$ such that $L \in \mathcal{A}$. If M is a feasible solution for (3.2), then its orthogonal projection onto the coherent algebra $\mathcal{A}$ generated by $\mathcal{C}$ is also feasible and has the same objective value. The same is true for any feasible solution N of (3.4).

Proof. Let M, N be feasible solutions to (3.2) and (3.4), respectively. By [7, Corollary 9.1], it follows that both projections M' and N' on $\mathcal{A}$ are positive semidefinite matrices.

As the basis matrices in $\mathcal{C}$ form an orthogonal basis to $\mathcal{A}$, we may write:

$$M' = \sum_{i=0}^{d} \frac{\langle M, A_i \rangle}{\langle A_i, A_i \rangle} \cdot A_i.$$

Now recall that

$$\langle M, A_i \rangle = \operatorname{tr}(M A_i^T) = \operatorname{sum}(M \circ A_i),$$

where $\circ$ denotes the entrywise matrix product—also known as the Schur product—and sum denotes the operator that adds all entries of a given matrix. If A_i is a fiber of the configuration, that is, a 01-diagonal matrix, and as the diagonal entries of M are all constant, it follows that $\operatorname{sum}(M \circ A_i)$ equals the number of 1s in A_i's diagonal, i.e., $\langle A_i, A_i \rangle$. Combining this with the previous expression for M' yields $\operatorname{diag}(M') = \mathbb{1}$, and the linearity of the operators involved also easily yields that $\operatorname{diag}(N') = \mu \cdot \mathbb{1}$. Since $L \in \mathcal{A}$ and the orthogonal projection operator is self-adjoint, it follows that

$$\langle M', L \rangle = \langle M, L' \rangle = \langle M, L \rangle,$$

hence M' has the same objective value as M.

It remains to show that $\mathcal{L}^*(N') \geq 4 \cdot \mathbb{1}$. First, since $L \in \mathcal{A}$ and L has a nonzero entry for each edge ij in G, it follows that there exists a unique basis matrix $A_l \in \mathcal{C}$ such that $(A_l)_{ij} = 1$, thus by expressing N' similarly to the formula for M' above, we obtain

$$N'_{ij} = \frac{\langle N, A_l \rangle}{\langle A_l, A_l \rangle} = \frac{\sum_{ab:(A_l)_{ab}=1} N_{ab}}{\langle A_l, A_l \rangle}.$$

As the basis matrices in $\mathcal{C}$ have disjoint support, it follows that all nonzero entries in A_l correspond to edges in G, and by the feasibility conditions of (3.4), it holds that, for any edge $ab \in E(G)$,

$$4 \leq \mathcal{L}^*(N)_{ab} = N_{aa} + N_{bb} - 2N_{ab} = 2\mu - 2N_{ab}.$$

Thus, for ij edge of G,

$$-N'_{ij} = \frac{\sum_{ab:(A_l)_{ab}=1} -N_{ab}}{\langle A_l, A_l \rangle} \geq 2 - \mu.$$

This in turn implies

$$\mathcal{L}^*(N')_{ij} = 2\mu - 2N'_{ij} \geq 4,$$

and as the objective value of (3.4) is given by μ, it follows that N' is feasible and has the same objective value as N, which concludes the proof. ■

The previous result allows us to assume that optimal solutions for (3.2) and (3.4) lie in $\mathcal{A}$ when it contains the Laplacian matrix L. For the remainder of this section, we will be interested in regular graphs that display similar properties to edge-transitive graphs. It is worth noting that for a k-regular graph, since $L = kI - A$, the condition that L belongs to a certain coherent algebra is equivalent to requiring that A also does. With these remarks, we can now conclude a type of equivalence between the feasibility conditions of the aforementioned programs.

Lemma 3.2.2. Let G be a graph with adjacency matrix A that either belongs to or splits in a coherent configuration $\mathcal{C}$, and let $M \in \mathcal{A}$. Then for any $\mu > 0$, $N := \mu \cdot M$ is a feasible solution for (3.4) if, and only if, M is a feasible solution for (3.2) with objective value at least $|E|/\mu$.

Proof. Let L be the Laplacian of G, $M \in \mathcal{A}$ and $\mu > 0$. Note that if A either belongs to or splits in $\mathcal{C}$, then the entries of M will be constant on the edges of G. This is immediate in the former case, and as for the latter case it suffices to note that if $A = A_i + A_i^T$ for some $A_i \in \mathcal{C}$, then

$$M_{ab} = \frac{\langle M, A_i \rangle}{\langle A_i, A_i \rangle} = \frac{\langle M, A_i^T \rangle}{\langle A_i^T, A_i^T \rangle} = M_{ba},$$

for any edge $ab \in E(G)$ such that $(A_i)_{ab} = 1$ — and similarly for the edges in A_i^T. Now, by noting that L can be seen as the weighted Laplacian function evaluated at $\mathbb{1}$, we get

$$\frac{1}{4}\langle L, M \rangle = \frac{1}{4}\langle \mathbb{1}, \mathcal{L}^*(M) \rangle = \frac{1}{4} \sum_{ij \in E(G)} \mathcal{L}^*(M)_{ij} = \frac{\mathcal{L}^*(M)_{ij} \cdot |E|}{4}, \tag{3.7}$$

for any $ij \in E(G)$, where the last equality follows from the previous remarks. If $N := \mu \cdot M$ is feasible for (3.4), then it easily follows that $M \succcurlyeq 0$ and $\operatorname{diag}(M) = \mathbb{1}$, and since $4 \leq \mathcal{L}^*(N)_{ij} = \mu \mathcal{L}^*(M)_{ij}$ for any $ij \in E(G)$, this combined with (3.7) yields that the objective value of M is at least $|E|/\mu$. Conversely, if M is feasible for (3.2) with objective value at least $|E|/\mu$, again it easily follows that $N \succcurlyeq 0, \operatorname{diag}(N) = \mu \cdot \mathbb{1}$, and combining the fact that all entries of M are constant on the edges of G with (3.7), we get that $\mathcal{L}^*(N)_{ij} \geq 4$ for any $ij \in E(G)$, which concludes the proof. ∎

Combining the two lemmas, we prove the main result of this section:

Theorem 3.2.3. If G is a graph whose adjacency matrix either belongs to or splits in a coherent configuration $\mathcal{C}$, then

$$\eta(G)\eta^\circ(G) = |E|.$$

Proof. Let M be an optimal solution for (3.2), which by Lemma 3.2.1 can be assumed to belong to the coherent algebra generated by $\mathcal{C}$. We can thus take $\mu := |E|/\eta > 0$ and consider the matrix $N := \mu \cdot M$. As M is feasible with objective value $\eta = |E|/\mu$, we can apply Lemma 3.2.2 to conclude that N is feasible for (3.4), with objective value $|E|/\eta$ by construction. Since (3.4) is a minimization problem, we get

$$\eta(G)\eta^\circ(G) \leq |E|,$$

which together with (3.6b) implies the desired result. ∎

We recall that if G is edge-transitive, and if Γ is its automorphism group, then the centralizer $C(\Gamma)$ is a coherent algebra whose coherent configuration either contains or splits A. In the case of distance-regular graphs, their Bose-Mesner algebra clearly contains A in its association scheme. Thus, we obtain the following corollary:

Corollary 3.2.4. If G is either an edge-transitive or a distance-regular graph, then (3.6b) is an equality.

■

3.2.2 Totally conformally rigid graphs

In Chapter 2, we established that totally conformally rigid graphs are either 1-walk-regular or 1-walk-biregular (Corollary 2.6.7). We can exploit this structural characterization to show that all totally conformally rigid graphs attain equality in (3.6b).

Let $\mathcal{A}$ be the adjacency algebra of a 1-walk-regular graph generated by its adjacency matrix A. This is a $*$-algebra, and orthogonal projection onto it preserves positive semidefiniteness. Furthermore, because G is 1-walk-regular, any matrix in $\mathcal{A}$ has constant diagonal entries and is constant on the edges of G. On the other hand, if G is 1-walk-biregular with bipartite partitions V_1 and V_2, we can construct a slightly broader $*$-algebra. Let I_1 and I_2 be the diagonal indicator matrices for these partitions. We define $\mathcal{A}$ as the algebra generated by $\{A, I_1, I_2\}$. The adjacency matrix naturally splits as $A = A_{12} + A_{21}$, where $A_{12} = I_1 A I_2 \in \mathcal{A}$ and $A_{21} = I_2 A I_1 \in \mathcal{A}$. Because the graph is 1-walk-biregular, the projection of any matrix onto $\mathcal{A}$ will be constant on the $V_1 \times V_1$ diagonals, constant on the $V_2 \times V_2$ diagonals, and constant across all edges of G.

In both cases, the Laplacian L belongs to the $*$-algebra $\mathcal{A}$, and the orthogonal projection of any feasible solution M for (3.2) onto $\mathcal{A}$ yields a positive semidefinite matrix M'. Because $I \in \mathcal{A}$ (in the biregular case, $I = I_1 + I_2$), the trace is preserved, ensuring that $\operatorname{diag}(M') = \mathbb{1}$. Consequently, analogous versions of Lemmas 3.2.1 and 3.2.2 hold, which promptly imply the following:

Corollary 3.2.5. If G is a totally conformally rigid graph, then (3.6b) is an equality.

■

3.2.3 Association schemes

If we consider a k-regular graph G with adjacency matrix A, it can be easily shown that

$$\eta(G) \le \frac{|V|}{4}(k - \lambda_{\min}(A)) \quad \text{and} \quad \eta^{\circ}(G) \ge \frac{2k}{k - \lambda_{\min}(A)}.$$

The first bound can be obtained by choosing an appropriate feasible solution to (3.2), and by combining this with (3.6b) one can promptly obtain the lower bound to η°. By further requiring that A belongs to an association scheme, Goemans and Rendl [37] proved that $\eta(G)$ is always equal to the aforementioned upper bound. In this section, we will show that, under the same assumptions, η° is also always equal to the above lower bound.

Throughout the remainder of this section, we assume that A belongs to an association scheme $\mathcal{S} = \{I, A, A_2, ..., A_d\}$ with Bose-Mesner algebra $\mathcal{A}$ and primitive idempotents $E_0, ..., E_d$ (see Section 1.2.1), and we let $P, Q \in M_{d+1}(\mathbb{R})$ be the first and second eigenmatrices of $\mathcal{S}$, respectively, that is, the matrices defined by the equations

$$A_i = \sum_{l=0}^{d} P_{li} \cdot E_l \quad \text{and} \quad E_i = \frac{1}{|V|} \sum_{l=0}^{d} Q_{li} \cdot A_l. \tag{3.8}$$

We note that $PQ = QP = |V| \cdot I$, and that these are both real matrices as $\mathcal{S}$ is a set of symmetric matrices. We denote the degrees of the scheme by $k_i = p^0_{ii}$, hence each matrix A_i can be seen as the adjacency matrix of a k_i-regular graph, and we further note that $P_{l0} = Q_{l0} = 1$, $P_{0l} = k_l, Q_{0l} = m_l, P_{ji} m_j = Q_{ij} k_i$, for all indices $0 \leq i, j, l \leq d$, where $m_l = \operatorname{tr}(E_l)$ (see [8, Ch.2] for more details). From these definitions one can obtain the well-known orthogonality relations:

$$\sum_{l=0}^{d} \frac{P_{il} P_{jl}}{k_l} = \begin{cases} \frac{|V|}{m_i} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \sum_{l=0}^{d} P_{li} P_{lj} m_l = \begin{cases} |V| k_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

From (3.8), the eigenvalues of any linear combination $\sum_i x_i \cdot A_i$ of the basis matrices of $\mathcal{A}$ are given by the linear combinations of the respective eigenvalues of the A_i, hence the constraint $M \in \mathcal{A}, M \succcurlyeq 0$ is equivalent to requiring that $Px \geq 0$, where $x \in \mathbb{R}^{d+1}$ is the vector of coefficients such that $M = \sum_i x_i \cdot A_i$. With these observations, we can prove the following:

Theorem 3.2.6. If G is a k-regular graph whose adjacency matrix A belongs to an association scheme, and if $\lambda_{\min}(A)$ is its smallest eigenvalue, then

$$\eta^\circ(G) = \frac{2k}{k - \lambda_{\min}(A)}.$$

In particular, we have

$$1 \leq \frac{\operatorname{fcc}(G)}{\frac{2k}{k - \lambda_{\min}(A)}} \leq \frac{1}{\alpha_{\mathrm{GW}}}.$$

Proof. The previous comments about association schemes combined with Lemma 3.2.1 allows us to rewrite the semidefinite program $\eta^\circ(G)$ as the linear program

$$\eta^\circ(G) = \min \left\{ x_0 : x \in \mathbb{R}^{d+1}, Px \geq 0, x_0 \geq 0, x_0 - x_1 \geq 2 \right\},$$

where $x = (x_0, ..., x_d)$, and the constraint $x_0 - x_1 \geq 2$ follows from the constraint $\mathcal{L}^*(N) \geq 4 \cdot \mathbb{1}$ and the fact that $\mathcal{L}^*(N)_{ij} = 2x_0 - 2x_1$ if we assume that $N \in \mathcal{A}$. We can then use linear program strong duality and some straightforward algebraic manipulation to obtain:

$$\eta^\circ(G) = \max \left\{ 2b : y \in \mathbb{R}_+^{d+1}, a, b \in \mathbb{R}_+, P^T y = b \cdot e_1 + (1 - a - b) \cdot e_0 \right\},$$

where e_i is the i-th canonical basis vector in $\mathbb{R}^{d+1}$, with indexing starting at 0. Noting that the inverse of P is $(1/|V|) \cdot Q$, that $P_{ji} m_j = Q_{ij} k_i$ and that the first row of Q contains the traces of the respective primitive idempotents, we get

$$y = \frac{1}{|V|} \sum_{l=0}^{d} \left(\left(\frac{bP_{l1}}{k} + 1 - a - b \right) m_l \right) \cdot e_l,$$

thus y is completely determined by our choice of a, b. We can also note that if (y, a, b) is feasible for our program, we can take $a' = 0$ and $y' = y + (a/|V|) \cdot Q^T e_0$, which is nonnegative since $Q_{0i} = m_i \geq 0$ for any index $0 \leq i \leq d$, and thus $(y', a' = 0, b)$ is a feasible solution with the same objective value. So we may always assume that an optimal solution has $a = 0$. Combining this with the fact that $P_{01} = k$ and that each entry of y must be nonnegative, we get that

$$b \leq \frac{k}{k - P_{l1}},$$

for all indices $1 \leq l \leq d$. As we wish to maximize b, we set

$$b = \min_{1 \leq l \leq d} \left\{ \frac{k}{k - P_{l1}} \right\} = \frac{k}{k - \lambda_{\min}(A)},$$

and with this the result follows immediately. ■

We remark that one could also obtain the previous result by combining Theorem 3.2.3 with the aforementioned result from [37] that $\eta(G) = (|V|/4)(k - \lambda_{\min}(A))$ for graphs in association schemes. However, the constructive proof given above highlights some methods that will be particularly useful in the next section.

3.3 Quadratic programs and MAX 2-SAT

We now study a broader framework for approximating quadratic programs via semidefinite programs, that generalizes the results for MAXCUT derived in the previous sections. Let $G_1 = (V, E_1), G_2 = (V, E_2)$ be two graphs on the same vertex set V, and let L be the Laplacian matrix of G_1 and K be the signless Laplacian matrix of G_2. We can define the quadratic program

$$\begin{aligned}\mathrm{qp}(G_1, G_2) &:= \max\left\{\sum_{ij\in E_1}(1 - x_i x_j) + \sum_{ij\in E_2}(1 + x_i x_j) : x \in \mathbb{R}^V, x_i^2 = 1\right\}\\ &= \max\left\{\left\langle \frac{L+K}{2}, xx^T\right\rangle : x \in \mathbb{R}^V, x_i^2 = 1\right\},\end{aligned} \tag{3.9}$$

with semidefinite relaxation given by

$$\gamma(G_1, G_2) := \max\left\{\left\langle \frac{L+K}{2}, M\right\rangle : M \succcurlyeq 0, \mathrm{diag}(M) = \mathbb{1}\right\}. \tag{3.10}$$

These programs can be seen as generalizations of MAXCUT and its semidefinite relaxation defined in equations (3.1) and (3.2), respectively, since

$$\mathrm{mc}(G_1) = \frac{1}{2}\mathrm{qp}(G_1, G_2) \quad \text{and} \quad \eta(G_1) = \frac{1}{2}\gamma(G_1, G_2),$$

if we set $E_2 = \emptyset$. The above programs can also be easily extended to allow for edge weights in G_1 and G_2, and in [38], Goemans and Williamson show how to use solutions of $\gamma(G_1, G_2)$ to construct an α_{GW}-approximation algorithm for $\mathrm{qp}(G_1, G_2)$.

Notably, (3.9) can be used to model MAX 2-SAT. Recall that an instance of the MAX 2-SAT problem consists of a collection of Boolean clauses $C_1, ..., C_m$, where each clause is a disjunction of at most two literals drawn from a set of variables $\{z_1, ..., z_n\}$, with a literal being either a variable or its negation. The problem asks then to assign truth values to the variables in order to maximize the number of satisfied clauses. This is a classic problem that is known to be NP-hard [30]. The approximation ratio of ≈ 0.878 given in [38] has been improved to ≈ 0.94 in general [52], with ratios as high as ≈ 0.954 attained for certain restricted versions of the problem [5, 16].

Following [38, Sec. 7.2.1], one can model MAX 2-SAT as a quadratic program by introducing $n+1$ variables $\{x_0, ..., x_n\}$ with $x_i \in \{\pm 1\}$, where x_0 is an auxiliary variable that encodes a truth assignment by setting z_i to true if $x_i = x_0$, and false otherwise, for $i \in [n]$. The value $v(C)$ of a clause C is defined to be 1 if it is true and 0 otherwise, and as clauses can have at most two literals, it is easy to manually check that the value of any clause can be expressed as a linear combination of terms of the form $1 + x_i x_j$ and $1 - x_i x_j$. The MAX 2-SAT problem can then be written as

$$\begin{aligned}&\max\left\{\sum_{j=1}^m v(C_j) : x_i^2 = 1, i \in [n]\right\}\\ &= \max\left\{\sum_{i,j\in\{0,...,n\}, i\neq j} \alpha_{ij}(1 - x_i x_j) + \beta_{ij}(1 + x_i x_j) : x_i^2 = 1, i \in [n]\right\},\end{aligned}$$

for some nonnegative coefficients α_{ij}, β_{ij}. The graphs G_1, G_2 can thus be constructed with vertex set given by the variables x_i and weighted edges defined by α_{ij}, β_{ij}, respectively, and similarly to what was done for MAXCUT, we study the instances in which all nonzero edge weights are constant.

Analogously to (3.4), define the program

$$\gamma^\circ(G_1,G_2) := \min\{\mu : \mu \geq 0, N \succcurlyeq 0, \operatorname{diag}(N) = \mu \cdot \mathbb{1}, \\ \frac{1}{2}\mathcal{L}^*(N) \geq \mathbb{1}, \frac{1}{2}\mathcal{K}^*(N) \geq \mathbb{1}\}, \tag{3.11}$$

where $\mathcal{K}^*$ is a function that maps a positive semidefinite matrix N to a vector in $\mathbb{R}_+^{E_2}$ such that $\mathcal{K}^*(N)_{ij} = N_{ii} + N_{jj} + 2N_{ij}$ — that is, the adjoint of the signless Laplacian function that maps nonnegative edge weights to the usual weighted signless Laplacian matrix. The parameters γ, γ° form a primal-dual pair of gauges (when specializing to the unweighted scenario, as discussed in the introduction) satisfying the inequality

$$|E_1| + |E_2| \leq \gamma(G_1,G_2)\gamma^\circ(G_1,G_2). \tag{3.12}$$

Similarly to what was done in the previous sections, we inquire into the optimal solutions of these semidefinite programs for graphs with high structural regularity, and also discuss the equality cases of (3.12).

From now on, we assume that the adjacency matrices A_1, A_2 of G_1, G_2 are elements of an association scheme $S = \{I, A_1, A_2, ..., A_d\}$ with Bose-Mesner algebra $\mathcal{A}$, and we use the same notation as in Section 3.2.3 to refer to the primitive idempotents and eigenmatrices of $\mathcal{S}$. We first note that the feasibility region of (3.10) is the same as (3.2), hence any feasible solution M may be assumed to belong to $\mathcal{A}$ by an analogous version of Lemma 3.2.1. We can apply a similar argument as in Section 3.2.3 to transform $\gamma(G_1,G_2)$ into a linear program: the constraint $\operatorname{Diag}(M) = \mathbb{1}$ implies that if $M \in \mathcal{A}$ then $M = I + \sum_{l=1}^d x_i \cdot A_i$, hence the constraint $M \succcurlyeq 0$ is equivalent to $Rx \geq 0$, where $x \in \mathbb{R}^d$ and R is the $(d+1) \times d$ matrix obtained by deleting the first column of P. Noting that $L = k_1 \cdot I - A_1, K = k_2 \cdot I + A_2$, it follows that

$$\langle \frac{L+K}{2}, M\rangle = \frac{1}{2}(\langle k_1 \cdot I - A_1, M\rangle + \langle k_2 \cdot I + A_2, M\rangle) = \frac{|V|}{2}((k_1+k_2) + (k_2x_2 - k_1x_1)),$$

and with this we can rewrite (3.10) as the following linear program:

$$\begin{aligned} \gamma(G_1,G_2) &= \frac{|V|}{2}((k_1+k_2) + \max\{(k_2 \cdot e_2 - k_1 \cdot e_1)^T x : x \in \mathbb{R}^d, Rx \geq -\mathbb{1}\}) \\ &= \frac{|V|}{2}((k_1+k_2) + \min\{\mathbb{1}^T y : y \in \mathbb{R}_+^{d+1}, R^T y = k_1 \cdot e_1 - k_2 \cdot e_2\}), \end{aligned} \tag{3.13}$$

where the second equality follows from strong duality. As a consequence, we explicitly compute $\gamma(G_1,G_2)$:

Theorem 3.3.1. If G_1, G_2 are graphs whose adjacency matrices A_1, A_2 belong to an association scheme with first eigenmatrix P, then

$$\gamma(G_1,G_2) = \frac{|V|}{2}\left((k_1+k_2) + \max_{0 \leq l \leq d}\{P_{l2} - P_{l1}\}\right).$$

Proof. We let z denote the first row of the second eigenmatrix Q, that is, we have that $z = (1, m_1, ..., m_d)$ is a vector containing the multiplicities of the scheme, and moreover, since $QP = |V| \cdot I$, and R is obtained from deleting the first column of P, it follows that $R^T z = 0$. As P is invertible, we get that R^T has rank d, which implies that $\ker(R^T) = \langle z \rangle$. On the other hand, we may use the orthogonality relations to explicitly construct a solution for the system $R^T y = k_1 \cdot e_1 - k_2 \cdot e_2$ by considering

$$y' = \frac{1}{|V|} \sum_{l=0}^{d} ((P_{l1} - P_{l2}) m_l) \cdot e_l,$$

and then noting that

$$(R^T y')_i = \frac{1}{|V|} \left(\sum_{l=0}^{d} P_{l1} P_{li} m_l - \sum_{l=0}^{d} P_{l2} P_{li} m_l \right) = \begin{cases} k_1, & \text{if } i = 1, \\ -k_2, & \text{if } i = 2, \\ 0, & \text{otherwise.} \end{cases}$$

This implies that any solution for $R^T y = k_1 \cdot e_1 - k_2 \cdot e_2$ is of the form $y = y' + (\alpha/|V|) \cdot z$, for any real α, but note also that $\mathbb{1}^T y' = 0$ by the orthogonality relations, and $\mathbb{1}^T z = \sum_{l=0}^{d} m_l = |V|$. Hence, the objective value of any feasible y is precisely α. We then need to choose the minimum α that makes y feasible, which is given by

$$\alpha^* = \max_{0 \le l \le d} \{P_{l2} - P_{l1}\},$$

since the entries of y must all be nonnegative, and this together with (3.13) concludes the proof. ■

As previously mentioned, one can construct an α_{GW}-approximation algorithm for $\mathrm{qp}(G_1, G_2)$ with the solutions of $\gamma(G_1, G_2)$, which combined with the previous theorem allows us to bound $\mathrm{qp}(G_1, G_2)$ in terms of the entries of the first eigenmatrix of the scheme. In particular, this also allows us to bound the value of MAX 2-SAT for the instances discussed at the beginning of this section.

Corollary 3.3.2. If G_1, G_2 are graphs whose adjacency matrices A_1, A_2 belong to an association scheme with first eigenmatrix P, then

$$\alpha_{\mathrm{GW}} \le \frac{\mathrm{qp}(G_1, G_2)}{\frac{|V|}{2} \cdot \left((k_1 + k_2) + \max_{0 \le l \le d} \{P_{l2} - P_{l1}\} \right)} \le 1.$$

■

We turn our attention to $\gamma^\circ(G_1, G_2)$, and in order to compute its optimal value analytically, we shall restrict ourselves to association schemes coming from distance-regular graphs. First, we recall that if a distance-regular graph G with diameter d has intersection array $\iota(G) = \{b_0, ..., b_{d-1}; c_1, ..., c_d\}$, and if P is the first eigenmatrix of the scheme generated by G, then

$$P_{l2} = \frac{k_2}{b_1 k_1}(P_{l1}^2 - (k_1 - b_1 - 1)P_{l1} - k_1), \tag{3.14}$$

for any index $0 \leq l \leq d$ (see [17, Ch.4] for more details). With this, we prove the following:

Theorem 3.3.3. Let G_1 be a distance-regular graph with diameter d and let G_2 be its distance-2 graph, with respective adjacency matrices A_1 and A_2. If P is the first eigenmatrix associated with the association scheme generated by A_1, then

$$\gamma^\circ(G_1, G_2) = \begin{cases} \frac{k_1}{k_1 - P_{d1}}, & \text{if } k_2 P_{d1} + k_1 P_{d2} > 0, \\ \frac{k_1}{k_1 - P_{d1}} - \frac{(k_2 P_{d1} + k_1 P_{d2})}{2k_2(k_1 - P_{d1})}, & \text{otherwise.} \end{cases}$$

Proof. Similarly to what was done in Theorem 3.2.6, we can rewrite $\gamma^\circ(G_1, G_2)$ as the following linear program:

$$\gamma^\circ(G_1, G_2) = \min\{x_0 : x \in \mathbb{R}^{d+1}, Px \geq 0, x_0 - x_1 \geq 1, x_0 + x_2 \geq 1\},$$

where the last two constraints follow from the fact that any feasible $N \in \mathcal{A}$ for (3.11) is constant on the edges of G_1 and G_2. By strong duality, we have

$$\begin{aligned} \gamma^\circ(G_1, G_2) = \max\{b + c : y \in \mathbb{R}_+^{d+1}, a, b, c \in \mathbb{R}_+, \\ P^T y = b \cdot e_1 - c \cdot e_2 + (1 - a - b - c) \cdot e_0\}, \end{aligned} \tag{3.15}$$

and since $|V|^{-1} \cdot Q$ is the inverse of P and $P_{ji} m_j = Q_{ij} k_i$, we get that

$$y = \frac{1}{|V|} \sum_{l=0}^{d} \left(\left(\frac{bP_{l1}}{k_1} - \frac{cP_{l2}}{k_2} + 1 - a - b - c \right) m_l \right) \cdot e_l,$$

and so y is determined by our choice of a, b, c. Similarly to the proof of Theorem 3.2.6, we may always assume that any optimal solution (y, a, b, c) is such that $a = 0$, and since each entry of y must be nonnegative, we get that $0 \leq c \leq 1/2$, and that

$$b \leq \min_{1 \leq l \leq d} \left\{ \left(\frac{k_1}{k_1 - P_{l1}} \right) \left(\frac{(1-c)k_2 - cP_{l2}}{k_2} \right) \right\}.$$

If we fix some feasible c, as we wish to maximize $b + c$, we may always assume that the previous inequality is an equality, and thus any solution to our program is in fact uniquely determined by our choice of c. A simple algebraic manipulation then shows that we can rewrite $\gamma^\circ(G_1, G_2)$ as

$$\gamma^\circ(G_1, G_2) = \max_{0 \leq c \leq 1/2} \left\{ \min_{1 \leq l \leq d} \left\{ \frac{k_1}{k_1 - P_{l1}} - c\frac{(k_1 P_{l2} + k_2 P_{l1})}{(k_1 - P_{l1})k_2} \right\} \right\}. \tag{3.16}$$

We now claim that for any fixed c, the index l that achieves the minimum in the previous equation is d, and we shall prove this by constructing a chain of equivalent inequalities. Indeed, for any index $1 \leq l \leq d$, define

$$\gamma_l^\circ(c) = \frac{k_1}{k_1 - P_{l1}} - c\frac{(k_1 P_{l2} + k_2 P_{l1})}{(k_1 - P_{l1})k_2},$$

and assume w.l.o.g. that the matrix P is ordered such that $P_{d1} < \ldots < P_{01}$, that is, we order the distinct eigenvalues of G_1 in decreasing order (and note that there are precisely $d+1$ of those as G_1 is distance-regular with diameter d). Now consider the indices $1 \leq j \leq i \leq d$, and note that it suffices to show that $\gamma_i^\circ(c) \leq \gamma_j^\circ(c)$ for a fixed $0 \leq c \leq 1/2$. A straightforward computation shows that this is equivalent to requiring that

$$P_{j2}(k_1 - P_{i1}) - P_{i2}(k_1 - P_{j1}) \leq k_2(P_{j1} - P_{i1}).$$

By substituting the terms P_{j2} and P_{i2} with the respective expressions given in (3.14) and noting that by assumption $P_{j1} - P_{i1} > 0$, we get that the previous inequality is equivalent to $P_{i1}(k_1 - P_{j1}) \leq k_1(k_1 - P_{j1})$, which holds since k_1 is the largest eigenvalue of G_1 and $j > 0$. By combining this fact with (3.16), we obtain

$$\gamma^\circ(G_1, G_2) = \max_{0 \leq c \leq 1/2} \left\{ \frac{k_1}{k_1 - P_{d1}} - c\left(\frac{k_2 P_{d1} + k_1 P_{d2}}{k_2(k_1 - P_{d1})}\right) \right\}.$$

If $k_2 P_{d1} + k_1 P_{d2} > 0$, we take $c = 0$ in order to maximize the objective value, and otherwise we take $c = 1/2$, which concludes the proof. ■

We remark that one could further reduce the expression obtained in the previous theorem for γ° by substituting P_{d2} according to (3.14), which would result in a term solely dependent on P_{d1}. This however results in a rather obscure expression that does not seem to present any further insight into the problem, and so we opted to omit it.

Since the semidefinite programs for γ and γ° generalize η and η°, a natural question to ask is whether the equality cases of (3.12) behave similarly to the ones of (3.6b). We have conducted computational experiments using SageMath [66] to explicitly check if (3.12) is an equality for a given pair of graphs G_1, G_2. In particular, we checked all strongly-regular graphs (SRGs) contained in SageMath's implementation of Brouwer's SRG database [21] with at most 1300 vertices. Of all 1058 SRGs in question, only 148 attained equality in (3.12), with the remaining 910 attaining a strict inequality. Notably, however, we managed to find large examples of SRGs for both cases, suggesting that equality in (3.12) might hold for more than a finite set of graphs. The smallest SRG where (3.12) is a strict inequality is the Paley graph of parameter 9, obtained by first considering a finite field $\mathbb{F}_9$ with 9 elements which will be the vertices of our graph, and then connecting two vertices if their difference is a square in $\mathbb{F}_9$. This is an arc-transitive graph and also a strongly-regular graph, hence its adjacency matrix A belongs to an association scheme $\{I, A, J - A - I\}$, with first eigenmatrix given by

$$P = \begin{pmatrix} 1 & 4 & 4 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}.$$

If we then let G_1, G_2 be such graph and its complement, respectively, one can easily see that the vector $y = (0, 1/2, 0, 0, 1/2, 1/4)$ is a feasible solution for the maximization program in (3.15) with objective value $3/4 > 8/11 = (|E_1| + |E_2|)/\gamma$, implying that $\gamma(G_1, G_2)\gamma^\circ(G_1, G_2) > |E_1| + |E_2|$.

Conclusion

In this thesis, we have investigated two interconnected themes in algebraic and spectral graph theory, both centered on the interplay between the algebraic structure of a graph and the behavior of optimization problems defined on it.

The first and main contribution of this thesis is the theory of totally conformally rigid graphs developed in Chapter 2. We introduced k-conformal rigidity as a generalization of the conformal rigidity of Steinerberger and Thomas [70], requiring the uniform weight assignment to simultaneously optimize the sum of the k largest and k smallest nontrivial Laplacian eigenvalues over all valid edge weightings. The strongest form of this property admits a remarkably complete characterization. Combining Corollaries 2.4.5, 2.5.5, and 2.6.7 with Theorem 2.8.3, we obtain the following seven-way equivalence:

Corollary 3.3.4. For a graph G, the following are equivalent:

(a) All edges of G are pairwise Laplacian-cospectral.

(b) G is edge-rigid.

(c) G is totally conformally rigid.

(d) For every Laplacian eigenprojector E_i, the vector $\mathcal{L}^*(E_i)$ is constant.

(e) For any orientation of the edges of G, the corresponding signed line graph of G is walk-regular.

(f) G is either 1-walk-regular or 1-walk-biregular.

(g) For all $k \in [n-1]$,
$$S_k(\mathbb{1})S_k^{\circ}(\mathbb{1}) = |E|.$$

■

These conditions connect a priori independent concepts from spectral geometry, combinatorial matrix theory, semidefinite optimization, and the theory of gauges into a unified picture. As an immediate consequence, we also showed that edge-rigidity can be decided in polynomial time. As combinatorial corollaries, we also showed via majorization theory that totally conformally rigid graphs maximize the number of spanning trees and minimize the Kirchhoff index over all valid edge weightings.

The second contribution, developed in Chapter 3, concerns semidefinite bounds for MAXCUT, the fractional cut-cover number, and MAX 2-SAT on graphs with high regularity.

We showed that the gauge duality equality $\eta(G)\eta^\circ(G) = |E|$ holds for all graphs whose adjacency matrix belongs to or splits in a coherent configuration, extending known results for edge-transitive graphs to the broader class of 1-walk-regular and 1-walk-biregular graphs. For graphs in association schemes, we computed $\eta^\circ(G)$ in closed form as a function of the smallest adjacency eigenvalue, yielding a spectral bound for FCC. In the broader MAX 2-SAT framework, we computed the optimal value of the semidefinite relaxation $\gamma(G_1, G_2)$ and its gauge dual $\gamma^\circ(G_1, G_2)$ explicitly in terms of the eigenmatrices of the underlying association scheme, with the latter further simplified for distance-regular graphs using the three-term recurrence for their eigenvalues. These computations yield spectral bounds for MAX 2-SAT and provide new insight into the relationship between the semidefinite programs and their combinatorial counterparts.

Several natural questions arise from the work presented here.

The most immediate open problem stemming from Chapter 2 is to understand the structure of graphs that are k-conformally rigid for some specific values of k, but not for all k simultaneously. Given any constant C, can one construct a graph that is k-conformally rigid for all $k \leq C$ but fails for $k = C + 1$? In particular, are there graphs that are 2-conformally rigid but not 1-conformally rigid? A related and more quantitative question is whether there exists a universal constant $c < 1$ such that if a graph is k-conformally rigid for all $k \leq cn$, then it is necessarily totally conformally rigid. Answering these questions would give a much finer picture of the hierarchy of conformal rigidity conditions. We also ask which graphs that are *not* totally conformally rigid still have $w = \mathbb{1}$ as the optimizer of $\tau(G, w)$ or $\mathrm{Kf}(G, w)$ over $w \in \Delta_E$, since total conformal rigidity is sufficient but may not be necessary for these extremal properties.

Regarding Chapter 3, we believe the results on coherent configurations and totally conformally rigid graphs from Sections 3.2.1 and 3.2.2 can provide further insight into the equality cases of the combinatorial inequality $|E| \leq \mathrm{mc}(G) \cdot \mathrm{fcc}(G)$, potentially leading to a deeper understanding of the relationship between MAXCUT and FCC. The example of the Paley graph on 9 vertices discussed in Section 3.3 shows that the analogous equality $|E_1| + |E_2| = \gamma(G_1, G_2)\gamma^\circ(G_1, G_2)$ can fail even for arc-transitive strongly regular graphs, suggesting that stronger conditions on the pair (G_1, G_2) are required. Characterizing the equality cases of (3.12) could lead to new interesting families of graphs and association schemes. Finally, determining the combinatorial interpretation of the gauge dual γ° of the MAX 2-SAT semidefinite relaxation, and investigating whether one can construct an approximation algorithm for the corresponding combinatorial parameter using γ°, are natural next steps that could yield further spectral bounds via the methods developed in this thesis.

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