Gauge duality for parameters of highly regular graphs

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ILAS 2026
Session: Spectral Graph Theory

May 23, 2026

Summary

Gauge Duality

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If $g(w) = 0$ implies $w = 0$, then g is *positive definite*, in which case a gauge is a norm in the nonnegative orthant. A gauge is called *monotone* if $g(w) \leq g(w')$ whenever $w \leq w'$ coordinate-wise.

Duality

A positive definite monotone gauge g admits a dual gauge, defined as

$$\begin{aligned} g^\circ(w) &= \max_{y \in \mathbb{R}^n_+} \{w^T y : g(y) \leq 1\} \\ &= \min_{\lambda \geq 0} \{\lambda : w^T y \leq \lambda g(y) \text{ for all } y \in \mathbb{R}^n_+\}. \end{aligned}$$

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$$w^T y \leq g(w) g^\circ(y). \qquad (1)$$

If g, g° are graph parameters, we will be interested in the following question: **for which graphs is the previous inequality an equality**?

Examples

Some examples of graph parameters that form primal-dual pairs of gauges:

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We will be interested in graphs $G = (V, E)$ **such that**:

$$\eta(G)\eta^\circ(G) = |E| \quad \text{or} \quad S_k(G)S_k^\circ(G) = |V|$$

FCC & MAX 2-SAT

Goemans-Williamson

Goemans and Williamson [GW95] famously shown that if we define

$$\eta(G,w) := \max\left\{\frac{1}{4}\langle\mathcal{L}(w),M\rangle : M \succcurlyeq 0, \operatorname{diag}(M) = \mathbb{1}\right\},$$

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then one can construct an $\alpha_{GW} \approx 0.878$ approximation algorithm for $\mathrm{mc}(G, w)$ such that

$$\alpha_{GW} \leq \frac{\eta(G, w)}{\mathrm{mc}(G, w)} \leq 1.$$

FCC

The gauge dual $\eta^\circ(G, z)$ is given by

$$\eta^\circ(G, z) := \min\{\mu : \mu \geq 0, N \succcurlyeq 0, \mathrm{diag}(N) = \mu \cdot \mathbb{1}, \frac{1}{4}\mathcal{L}^*(N) \geq z\}$$

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It was recently shown [PdCSST25] that one can construct an $1/\alpha_{GW}$-approximation algorithm for $\mathrm{fcc}(G, z)$ such that

$$1 \leq \frac{\eta^\circ(G, z)}{\mathrm{fcc}(G, z)} \leq \frac{1}{\alpha_{GW}}.$$

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Theorem 1 (A.,Coutinho [AC26])

If G is a graph that is 1-walk-regular or whose adjacency matrix either belongs to or splits in a coherent configuration $\mathcal{C}$, that is, either $A \in \mathcal{C}$ or $A = A_i + A_i^T$ for some $A_i \in \mathcal{C}$, then

$$\eta(G)\eta^{\circ}(G) = |E|.$$

In particular, this includes all edge-transitive and distance-regular graphs.

Association schemes

- If G is k-regular, it is known that

$$\eta(G) \leq \frac{|V|}{4}(k - \lambda_{\min}(A)) \quad \text{and} \quad \eta^\circ(G) \geq \frac{2k}{k - \lambda_{\min}(A)}.$$

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- Spectral bounds can also be obtained for MAX 2-SAT for graphs in association schemes (and in particular distance-regular graphs) by analyzing SDPs used to approximate generalizations of $\mathrm{mc}(G)$ [AC26].

Total Conformal Rigidity

Conformal Rigidity

- Let $S_k(w)$ be the sum of the k-largest eigenvalues of $\mathcal{L}(w)$ $(0 = \lambda_1(w) \leq \ldots \leq \lambda_n(w))$. We also let $s_k(w)$ denote the sum of the k smallest nontrivial eigenvalues of $\mathcal{L}(w)$.

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- **Which graphs satisfy** $S_k(\mathbb{1})S_k^{\circ}(\mathbb{1}) = |E|$?
- G is *conformally rigid* if $\lambda_2(w) \leq \lambda_2(\mathbb{1}) \leq \lambda_n(\mathbb{1}) \leq \lambda_n(w')$, for all $w, w' \in \mathbb{R}_+^E, \mathbb{1}^T w = \mathbb{1}^T w' = |E|$ [ST25].

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- All edge-transitive and 1-walk-regular graphs are conformally rigid [GST25].
- We say that G is *totally conformally rigid* if $s_k(w) \leq s_k(\mathbb{1}) \leq S_k(\mathbb{1}) \leq S_k(w')$ for all k.

Laplacian Cospectrality

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Theorem 2 (A.,Coutinho,Godsil [ACG26])

Let G be an undirected, simple, connected graph. Then G is totally conformally rigid if, and only if, all edges of G are pairwise cospectral.

Walk-regularity

- Recall that a graph is said to be *walk-regular* if, for any $l \geq 0$, the number of closed walks of length l centered at an vertex $a \in V$ is a constant that only depends on l.

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- G is said to be 1-walk-biregular if it is biregular bipartite, and the aforementioned constants also depend on which partition of V the vertices are located.

Theorem 3 (A., Coutinho, Godsil [ACG26])

A graph G is totally conformally rigid if, and only if, it is either 1*-walk-regular or* 1*-walk-biregular.*

Further characterizations

- $S_k(w)$ is a positive definite monotone gauge, and one can easily define its gauge dual $S_k^{\circ}(z)$. It can be shown that $S_k(\mathbb{1})S_k^{\circ}(\mathbb{1}) = |E|$ if, and only if, G is totally conformally rigid.

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- We can also characterize total conformal rigidity in terms of embeddings: the condition is equivalent to all canonical spectral embeddings onto the eigenspaces of L being edge-isometric.
- Lastly, we can also show that this is equivalent to the following: for any orientation of G, the corresponding signed line graph is walk-regular.

Full characterization

Theorem 4 (A., Coutinho, Godsil [ACG26])

For a graph G, the following are equivalent:

1. *All edges of G are pairwise Laplacian-cospectral.*
2. *G is edge-rigid.*
3. *G is totally conformally rigid.*
4. *For every Laplacian eigenprojector E_i, the vector $\mathcal{L}^*(E_i)$ is constant.*
5. *For any orientation of the edges of G, the corresponding signed line graph of G is walk-regular.*
6. *G is either 1-walk-regular or 1-walk-biregular.*
7. *For all $k \in [n-1]$,*

$$S_k(\mathbb{1})S_k^\circ(\mathbb{1}) = |E|.$$

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Questions?

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