Gauge Duality and Maxcut

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Overview

Sign-invariant norms: basic definitions

Let V be a finite set, and let $||\cdot||$ be a norm on $\mathbb{R}^V$. Recall the following definitions:

- $\mathbb{B}$ denotes the **unit ball** associated with $||\cdot||$;
- $||y||^* = \max_{x\in\mathbb{B}} x^T y$ denotes the **dual norm** of $||\cdot||$;
- $\mathbb{B}^\circ = \{y \in \mathbb{R}^V | x^T y \leq 1, \forall x \in \mathbb{B}\}$ denotes the **polar ball** w.r.t. $\mathbb{B}$;
- $\gamma_{\mathbb{B}}(x) = \inf\{\mu \in \mathbb{R}_+ | x \in \mu\mathbb{B}\}$ denotes the **Minkowski functional** w.r.t. $\mathbb{B}$.

We say that a norm is **sign-invariant** if $||x|| = |||x|||$ for any $x \in \mathbb{R}^V$, and we say that a set $\mathcal{X} \subseteq \mathbb{R}^V$ is **sign-symmetric** if $x \in \mathcal{X}$ iff $|x| \in \mathcal{X}$, similarly for any $x \in \mathbb{R}^V$.

Sign-invariant norms: construction

Proposition 1 (Constructing a sign-invariant norm)

If $\mathbb{B} \subseteq \mathbb{R}^V$ is a sign-symmetric, compact, convex set, with $0 \in \text{int}(\mathbb{B})$ and such that $\mathbb{B} = -\mathbb{B}$, then:

1. The Minkowski functional $\gamma_{\mathbb{B}}$ is a sign-invariant norm with unit ball given by $\mathbb{B}$;
2. The dual norm $||y||^* = \max_{x \in \mathbb{B}} x^T y$ is a sign-invariant norm with unit ball given by $\mathbb{B}^\circ$.

Sign-invariant norms: duality

Proposition 2 (Duality of sign-invariant norms)

If $||\cdot||$ is a sign-invariant norm on $\mathbb{R}^V$, with unit ball $\mathbb{B}$, then:

1. $\mathbb{B}$ is a sign-symmetric, compact convex set with 0 in its interior, and $\mathbb{B} = -\mathbb{B}$;
2. $||\cdot||^*$ is a sign-invariant norm with unit ball $\mathbb{B}^\circ$;
3. $||\cdot||^{**} = ||\cdot||$, and $\mathbb{B}^{\circ\circ} = \mathbb{B}$;
4. For every $x, y \in \mathbb{R}^V$, we have

$$x^T y \leq ||x|| \cdot ||y||^*$$

In other words, any sign-invariant norm $||\cdot||$ can be expressed as:

$$||x|| = \max\{x^T y | y \in \mathbb{B}^\circ\} = \max\{x^T y | y \in \mathbb{R}^V, ||y||^* \leq 1\}$$

Convex corners and antiblockers

Let $\mathcal{X} \subseteq \mathbb{R}_+^V$ be a subset of the nonnegative orthant, and consider the following definitions:

- We say that $\mathcal{X}$ is **lower-comprehensive** if for any $y \in \mathcal{X}$, and for any $0 \le x \le y$, we have $x \in \mathcal{X}$;
- A **convex corner** is a lower-comprehensive compact convex set with nonempty interior that lies in the nonnegative orthant $\mathbb{R}_+^V$;
- We define the **antiblocker** of $\mathcal{X}$ as

$$\text{abl}(\mathcal{X}) := \mathcal{X}^\circ \cap \mathbb{R}_+^V = \{y \in \mathbb{R}_+^V | x^T y \le 1, \forall x \in \mathcal{X}\}$$

We can also define the **lower-comprehensive hull** $\text{lc}(\mathcal{X})$ of a set $\mathcal{X}$ as the intersection of all lower-comprehensive sets that contain $\mathcal{X}$.

Gauges: basic definitions

We say that a function $\kappa : \mathbb{R}_+^V \mapsto \mathbb{R}$ is a **gauge** if:

1. κ is **positive semidefinite**, that is, $\kappa(w) \geq 0$ for every $w \in \mathbb{R}_+^V$, and $\kappa(0) = 0$;
2. κ is **positively homogeneous**, that is, $\kappa(\lambda w) = \lambda \kappa(w)$ for every $\lambda > 0$ and $w \in \mathbb{R}_+^V$;
3. κ is **sublinear**, that is, $\kappa(w + z) \leq \kappa(w) + \kappa(z)$, for every $w, z \in \mathbb{R}_+^V$.

We say that a gauge is **positive definite** if $\kappa(w) > 0$ whenever $w \in \mathbb{R}_+^V$ and $w \neq 0$, and **monotone** if $\kappa(w) \leq \kappa(z)$ whenever $w \leq z$.

Gauges and sign-invariant norms

Lemma 3

Let $||\cdot||$ be a sign-invariant norm on $\mathbb{R}^V$, and let κ be a positive definite monotone gauge. Then:

1. *the restriction of $||\cdot||$ to $\mathbb{R}^V_+$ is a positive definite monotone gauge;*
2. *The function $||\cdot||_\kappa$ that maps $x \in \mathbb{R}^V$ to $\kappa(|x|)$ is a sign-invariant norm.*

Given a sign-invariant norm $||\cdot||$, we say that the set $\mathcal{C} = \mathbb{B} \cap \mathbb{R}^V_+$ is its **unit convex corner**. The unit convex corner of a positive definite monotone gauge is the unit convex corner of the norm $||\cdot||_\kappa$ described in the previous Lemma.

Gauges: construction

Theorem 4 (Construction of Gauges)

Let $\mathcal{C} \subseteq \mathbb{R}_+^V$ be a convex corner. Then:

1. *the restriction of the Minkowski functional $\gamma_{\mathcal{C}}$ to $\mathbb{R}_+^V$ is a positive definite monotone gauge with unit convex corner $\mathcal{C}$;*
2. *The restriction of the dual norm $||y||^* = \max_{x \in \mathcal{C}} x^T y$ to $\mathbb{R}_+^V$ is a positive definite monotone gauge with unit convex corner $abl(\mathcal{C})$.*

If κ is a positive definite monotone gauge, then the **dual gauge** κ° is given by

$$\kappa^\circ(z) = \max\{w^T z | w \in \mathbb{R}_+^V, \kappa(w) \leq 1\}$$

Gauges: Duality

Theorem 5

If κ is a positive definite monotone gauge, $\mathbb{B}$ is the unit ball of the sign-invariant norm $||\cdot||_\kappa$, and $\mathcal{C} = \mathbb{B} \cap \mathbb{R}_+^V$, then:

1. *$\mathcal{C}$ is a convex corner;*
2. *κ° is a positive definite monotone gauge with unit convex corner abl($\mathcal{C}$);*
3. *$\kappa^{\circ\circ} = \kappa$, and abl(abl($\mathcal{C}$)) $= \mathcal{C}$;*
4. *For every $w, z \in \mathbb{R}_+^V$, we have*

$$w^T z \leq \kappa(w)\kappa^\circ(z)$$

Gauges: Putting it all together

- If κ is a positive definite monotone gauge, then the associated unit convex corner is $\mathcal{C}_\kappa = \{w \in \mathbb{R}_+^V | \kappa(w) \leq 1\}$. If $||\cdot||_\kappa$ is the induced sign-invariant norm, its unit ball is $\mathbb{B}$ s.t. $\mathcal{C}_\kappa = \mathbb{B} \cap \mathbb{R}_+^V$. The dual norm $||\cdot||_\kappa^*$ is also sign-invariant, and has unit ball given by the polar $\mathbb{B}^\circ$. Its restriction to $\mathbb{R}_+^V$ induces the dual positive definite monotone gauge κ°, with unit convex corner given by $\mathbb{B}^\circ \cap \mathbb{R}_+^V = \text{abl}(\mathcal{C}_\kappa)$. By gauge duality, we obtain:

$$\kappa(w) = \max\{z^T w | z \in \mathcal{C}_{\kappa^\circ}\} = \max\{z^T w | z \in \text{abl}(\mathcal{C}_\kappa)\}$$

- On the other hand, if $\mathcal{C}$ is a convex corner, we can obtain a positive definite monotone gauge whose unit convex corner is $\mathcal{C}$ by taking the restriction of the Minkowski functional $\gamma_\mathcal{C}$ to $\mathbb{R}_+^V$, and similarly the dual norm w.r.t. $\mathcal{C}$ will induce a positive definite monotone gauge whose unit convex corner is $\text{abl}(\mathcal{C})$.

Gauge duality: familiar example

Recall the definitions of the following sets:

$$\text{STAB}(G) := \text{conv}(\{\mathbf{1}_S \in \mathbb{R}^V | S \text{ is a coclique of } G\})$$
$$\text{QSTAB}(G) := \{x \in \mathbb{R}^V_+ | \mathbf{1}_K^T x \leq 1, \text{for every clique } K \text{ of } G\}$$

We then define the parameters:

$$\alpha(G, w) = \max\{w^T x | x \in \text{STAB}(G)\}$$
$$\chi_f(G, w) = \max\{w^T x | x \in \text{QSTAB}(\overline{G})\}$$

where α is the weighted maximum stable set of G, and χ_f is the weighted fractional chromatic number of G.

Gauge duality: familiar example

It can be shown that:

- $\alpha(G,w), \chi_f(G,w)$ are a dual pair of positive definite monotone gauges, that is, $\alpha(G,w)^\circ = \chi_f(G,w)$ and $\chi_f(G,w)^\circ = \alpha(G,w)$, and also:

$$\text{abl}(\text{QSTAB}(\overline{G})) = \text{STAB}(G)$$
$$\text{abl}(\text{STAB}(G)) = \text{QSTAB}(\overline{G})$$

- The unit convex corner of $\alpha(G,w)$ is $\text{QSTAB}(\overline{G})$, and similarly the unit convex corner of $\chi_f(G,w)$ is $\text{STAB}(G)$.

Maxcut and Fractional cut-cover

Let $G = (V, E)$ be a simple graph. Then:

- For every $S \subseteq V$, we denote by $\delta(S) \subseteq E$ as the cut induced by S, i.e., the set of all edges in G with precisely one endpoint in S;
- If $w \in \mathbb{R}_+^E$ is a vector with edge-weights, we define the **maximum cut** of G w.r.t. w as

$$\mathrm{mc}(G, w) := \max\{w^T \mathbf{1}_{\delta(S)} | S \subseteq V\}$$

and we similarly define the **fractional cut-cover** of G w.r.t $z \in \mathbb{R}_+^E$ as

$$\mathrm{fcc}(G, z) := \min\{\mathbf{1}^T y | y \in \mathbb{R}_+^{\mathcal{P}(V)}, \sum_{S \subseteq V} y_S \mathbf{1}_{\delta(S)} \geq z\}$$

SDPs

Recall the Goemanns-Williamson SDP for approximating $\mathrm{mc}(G, w)$:

$$\eta(G, w) := \max\{\langle \frac{\mathcal{L}(w)}{4}, X \rangle | X \succcurlyeq 0, \mathrm{diag}(X) = \mathbf{1}\}$$
$$= \min\{\mathbf{1}^T x | x \in \mathbb{R}^V, \mathrm{Diag}(x) \succcurlyeq \frac{\mathcal{L}(w)}{4}\}$$

We know that

$$\alpha_{GW} \eta(G, w) \leq \mathrm{mc}(G, w) \leq \eta(G, w)$$

where $\alpha_{GW} \approx 0.878$.

SDPs

We can define the following SDP:

$$\eta^\circ(G,z) := \min\{\mu | \mu \in \mathbb{R}_+, \, Y \succcurlyeq 0, \mathrm{diag}(Y) = \mu \mathbf{1}, \frac{\mathcal{L}^*(Y)}{4} \geq z\}$$
$$= \max\{w^T z | w \in \mathbb{R}_+^E, x \in \mathbb{R}^V, \mathrm{Diag}(x) \succcurlyeq \frac{\mathcal{L}(w)}{4}, \mathbf{1}^T x \leq 1\}$$

It can be shown that:

$$\eta^\circ(G,z) \leq \mathrm{fcc}(G,z) \leq \frac{1}{\alpha_{GW}} \eta^\circ(G,z)$$

where $1/\alpha_{GW} \approx 1.138$. Our goal now is to show that $\mathrm{mc}(G,w)$ and $\mathrm{fcc}(G,z)$ are a pair o dual positive definite monotone gauges, and also to show that the same is true for $\eta(G,w)$ and $\eta^\circ(G,z)$.

Applying gauge duality

Proposition 6

If $G = (V, E)$ is a graph, then the functions $\mathrm{mc}(G, w)$ and $\mathrm{fcc}(G, z)$ satisfy:

$$\mathrm{mc}(G, w) = \max\{w^T z | z \in \mathbb{R}^E_+, \mathrm{fcc}(G, z) \le 1\}$$
$$\mathrm{fcc}(G, z) = \max\{z^T w | w \in \mathbb{R}^E_+, \mathrm{mc}(G, w) \le 1\}$$

Moreover, the functions $\eta(G, w)$ and $\eta^\circ(G, z)$ satisfy:

$$\eta(G, w) = \max\{w^T z | z \in \mathbb{R}^E_+, \eta^\circ(G, z) \le 1\}$$
$$\eta^\circ(G, z) = \max\{z^T w | w \in \mathbb{R}^E_+, \eta(G, w) \le 1\}$$

Finding the convex corners

From the previous result, it follows that the functions $\mathrm{mc}(G, w)$ and $\eta(G, w)$ are positive definite monotone gauges, and their respective gauge duals are $\mathrm{fcc}(G, z)$ and $\eta^\circ(G, z)$. Define:

$$\mathrm{CUT}(G) := \mathrm{conv}(\{\mathbf{1}_{\delta(S)|S \subseteq V}\}) \subseteq \mathbb{R}_+^E$$
$$\mathrm{CUT}_{\mathrm{SDP}}(G) := \{\frac{\mathcal{L}^*(Y)}{4} | Y \succcurlyeq 0, \mathrm{diag}(Y) = \mathbf{1}\} \subseteq \mathbb{R}_+^E$$

Both sets are convex, compact and with non-empty interior. They are not, however, lower-comprehensive, and so we consider the sets $\mathrm{lc}(\mathrm{CUT}(G)), \mathrm{lc}(\mathrm{CUT}_{\mathrm{SDP}}(G))$.

Finding the convex corners

During the proof of the previous proposition, we've implicitly shown that

$$\begin{aligned} z \in \text{lc}(\text{CUT}(G)) &\iff \text{fcc}(G, z) \leq 1 \\ w \in \text{abl}(\text{CUT}(G)) &\iff \text{mc}(G, w) \leq 1 \\ z \in \text{lc}(\text{CUT}_{\text{SDP}}(G)) &\iff \eta^{\circ}(G, z) \leq 1 \\ w \in \text{abl}(\text{CUT}_{\text{SDP}}(G)) &\iff \eta(G, w) \leq 1 \end{aligned}$$

Hence the unit convex corners are

$$\begin{aligned} \mathcal{C}_{\text{fcc}} = \text{lc}(\text{CUT}(G)) \quad &\text{and} \quad \mathcal{C}_{\text{mc}} = \text{abl}(\text{CUT}(G)) \\ \mathcal{C}_{\eta^{\circ}} = \text{lc}(\text{CUT}_{\text{SDP}}(G)) \quad &\text{and} \quad \mathcal{C}_{\eta} = \text{abl}(\text{CUT}_{\text{SDP}}(G)) \end{aligned}$$

Putting it all together

We can then finally conclude that:

$$
\begin{aligned}
\mathrm{mc}(G, w) &= \max\{w^T z | z \in \mathrm{lc}(\mathrm{CUT}(G))\} \\
\mathrm{fcc}(G, z) &= \max\{z^T w | w \in \mathrm{abl}(\mathrm{CUT}(G))\} \\
\eta(G, w) &= \max\{w^T z | z \in \mathrm{lc}(\mathrm{CUT}_{\mathrm{SDP}}(G))\} \\
\eta^\circ(G, z) &= \max\{z^T w | w \in \mathrm{abl}(\mathrm{CUT}_{\mathrm{SDP}}(G))\}
\end{aligned}
$$

Concluding thoughts

There are numerous other interesting results related to $\mathrm{mc}(G, w), \mathrm{fcc}(G, z), \eta(G, w), \eta^\circ(G, z)$:

- The authors of [1] show how to extend the Goemans-Williamson algorithm for computing $\mathrm{fcc}(G, z)$;
- It can be shown that if G is edge-transitive, then $\mathrm{mc}(G, \mathbf{1})\mathrm{fcc}(G, \mathbf{1}) = m$;
- It can also be shown that if G belongs to a coherent algebra that contains $A(G)$ in its Schur basis, then

$$\eta(G, \mathbf{1})\eta^\circ(G, \mathbf{1}) = m$$

  e.g., in the case of distance-regular and edge-transitive graphs (and I'm 99% sure this is also true for 1-walk-regular graphs).

Bibliography

[1] N. B. Proença, M. K. de Carli Silva, C. M. Sato, and L. Tunçel, *A primal-dual extension of the goemans–williamson algorithm for the weighted fractional cut-covering problem*, 2023. arXiv: 2311.15346 [math.OC]. [Online]. Available: https://arxiv.org/abs/2311.15346.

[2] N. B. Proença, M. K. de Carli Silva, and G. Coutinho, *Dual hoffman bounds for the stability and chromatic numbers based on sdp*, 2020. arXiv: 1910.05586 [math.CO]. [Online]. Available: https://arxiv.org/abs/1910.05586.

Questions?