CNMAC 2024
XLIII Congresso Nacional de Matemática Aplicada e Computacional

UFMG

Primal-dual inequalities for the approximate maximum-cut value of Graphs in Homogeneous Coherent Configurations

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Abstract

We show that the product between the Goemans-Williamson approximation for MAX-CUT and its gauge dual is upper bounded by the number of edges of the graph, for graphs whose adjacency matrix belong to certain $*$-algebras. Our work extends this inequality, which was previously known to hold for edge-transitive graphs, to include distance-regular graphs and 1-walk-regular graphs.

Keywords: gauge duality;semidefinite optimization;algebraic graph theory

Introduction

In their seminal paper [2] in semidefine optimization, Goemans and Williamson introduced a semidefinite program $\eta(G)$ for approximating the maximum-cut $\mathrm{mc}(G)$ of a simple undirected graph G, proving that $\eta(G)$ approximates $\mathrm{mc}(G)$ with a factor of ≈ 0.878. The maximum-cut parameter is also closely related to the fractional cut-cover parameter $\mathrm{fcc}(G)$ by the inequality

$$\mathrm{mc}(G)\mathrm{fcc}(G) \geq m, \tag{1}$$

where m is the number of edges of G, and it can be shown that equality holds if G is edge-transitive. The authors of [3] studied the relationships between these parameters under the general framework of Gauge Duality [4], and introduced a semidefinite program $\eta^{\circ}(G)$ that approximates $\mathrm{fcc}(G)$ with a factor of $\approx 1/0.878 = 1.138$, and that satisfies

$$\eta(G)\eta^{\circ}(G) \geq m, \tag{2}$$

where equality also holds for edge-transtive graphs.

We say that a subset $\mathcal{A}$ of the set $M_n(\mathbb{C})$ of $n \times n$ complex matrices is a $*$-**algebra** if it is a $\mathbb{C}$-subalgebra that is closed under the conjugate-transpose map. If π is the orthogonal projection onto $\mathcal{A}$, we say that it **preserves constant diagonal entries** if π maps matrices with constant diagonal entries in $M_n(\mathbb{C})$ to matrices with same constant diagonal entries in $\mathcal{A}$.

By a **coherent algebra** we mean a $*$-subalgebra of $M_n(\mathbb{C})$ that is also closed under the Schur product $\circ$, i.e., the elementwise matrix product. It can be shown that these properties guarantee the existence of a $\mathbb{C}$-basis $A_0, A_1, ..., A_d$ of 01-matrices for $\mathcal{A}$ such that $\sum_{i=0}^{d} A_i = J$, where J is the all-ones matrix, and this basis is referred to as the **coherent configuration** (or **Schur basis**) associated with $\mathcal{A}$. A graph is said to belong to a coherent algebra if its adjacency matrix can be written as a sum of elements in the associated coherent configuration.

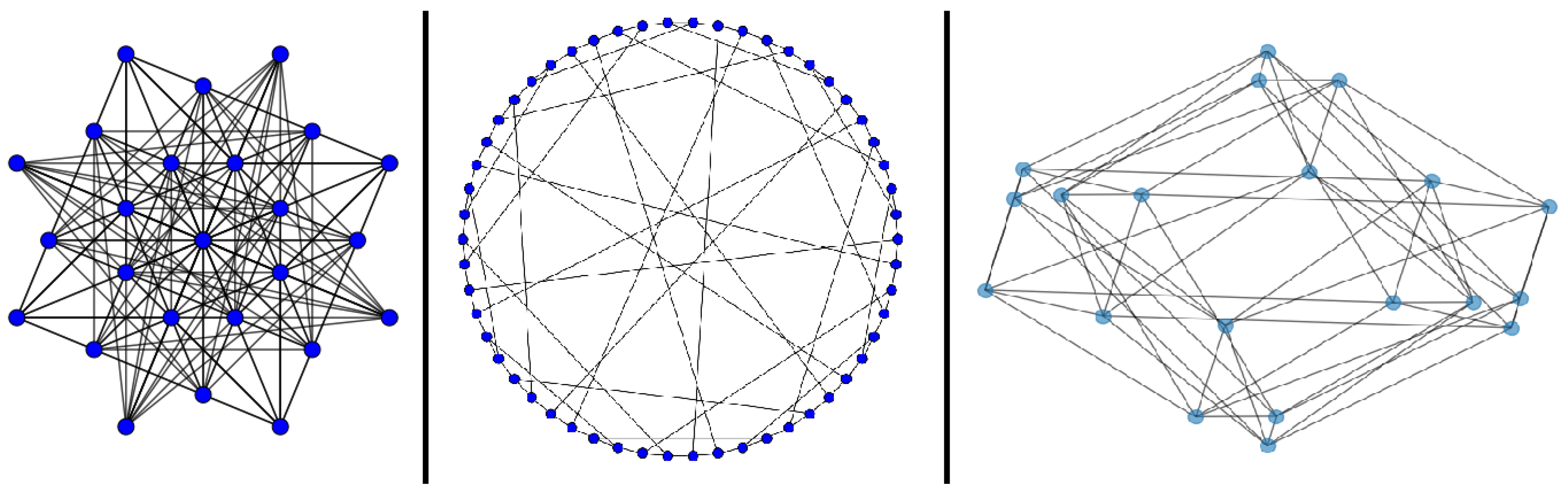

Figure 1: Examples of distance-regular, edge-transitive and 1-walk-regular graphs, respectively.

Objectives

This work aims to answer the following questions:

- Can we characterize the equality cases of Eq.1 and Eq.2 in terms of well-known families of graphs?
- Are there families of graphs for which Eq.2 is always an equality but Eq.1 is always a strict inequality?

Results

Our main result gives a general sufficient condition for attaining equality in Eq.2.

Theorem 1. *Let G be a simple undirected graph with n vertices and m edges with adjacency matrix A, and let $\mathcal{A}$ be a $*$-algebra with an orthogonal basis $\{A_0, A_1 = A, ..., A_d\}$ containing A such that $A_i \circ A = 0$ for all $i \neq 1$, and whose orthogonal projection preserves constant diagonal entries. Then*

$$\eta(G)\eta^{\circ}(G) = m.$$

This theorem allows us to prove the desired equality for well-known families of graphs.

Corollary 2. *If G is a graph that is either 1-walk-regular or that belongs to a coherent algebra that contains A in its coherent configuration, then*

$$\eta(G)\eta^{\circ}(G) = m.$$

In particular, the equality holds for all distance-regular and edge-transitive graphs.

In the case of distance regular graphs, the distance-partition of the graph – that is, the set of adjacency matrices of all k-distance graphs obtained from G – spans a coherent algebra that satisfies the desired properties. For edge-transitive graphs, the commutant algebra of the automorphism group is also a coherent algebra that satisfies the required conditions. Finally, even though not all 1-walk-regular graphs belong to coherent algebras, the algebra generated by their adjacency matrix is a $*$-algebra that satisfies the requirements of Theorem 1.

Conclusions

In this work, we extended a previously known inequality to include more general families of regular graphs. Our results also provide directions for further study on the equality cases of Eq.1 and Eq.2

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Acknowledgements

All authors acknowledge the support of CNPq and FAPEMIG.