MAXCUT, Association Schemes and Semidefinite Programs

Henrique Assumpção

Joint work with Gabriel Coutinho

Universidade Federal de Minas Gerais

henriquesoares@dcc.ufmg.br

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Summary

MAXCUT & SDPs

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3. $\mathbb{R}^V$ is the set of $|V| = n$ dimensional real-vectors indexed by V.

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- Hence we obtain:

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- The last constraint is the problematic one.

Approximating MAXCUT via SDPs

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- Goemans and Williamson [GW95] proved that:

$$\alpha_{\mathsf{GW}} \eta(G) \leq \mathrm{mc}(G) \leq \eta(G),$$

where $\alpha_{\mathsf{GW}} \approx 0.878$. This factor is optimal assuming the UGC and P $\neq$ NP.

Duality

- A problem dual to MAXCUT is the Fractional cut-cover (FCC) problem:

$$\mathrm{fcc}(G) := \min \left\{ \mathbb{1}^T y : y \in \mathbb{R}_+^{\mathcal{P}(V)}, \sum_{S \subseteq V} y_S \cdot \mathbb{1}_{\delta(S)} \geq \mathbb{1} \right\}, \quad (1)$$

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- [BPdCSST25] show how to use $\eta^\circ(G)$ to obtain a $1/\alpha_{\mathrm{GW}}$ approximation algorithm for FCC.

Association Schemes

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$$A_i A_j = \sum_{l=0}^{d} p_{ij}^l \cdot A_l,$$

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     for any indices $0 \leq i, j \leq d$.
- If $I \in \mathcal{C}$ and each A_i is symmetric, we say that $\mathcal{C}$ is an *association scheme*.

Algebras and projections

- Condition (4) is equivalent to requiring that $\mathcal{M} := \text{span}_{\mathbb{C}}(A_0, \dots, A_d)$ is an *algebra* over $\mathbb{C}$. In the case of association schemes, this will be a commutative $*$-algebra, and hence diagonalizable.

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- [BGSV12] shows that if M is a PSD matrix, then its orthogonal projection M' onto a $*$-algebra is also PSD.
- This allows us to project the feasible region of certain SDPs onto the $*$-algebras associated with highly regular graphs, which allows for the use of many powerful algebraic tools.

Bounds on FCC and MAX 2-SAT

General strategy

- If we restrict ourselves to graphs whose adjacency matrices belong to certain $*$-algebras (e.g. distance-regular graphs), we can easily show that the optimal solutions for the parameters $\eta(G), \eta^{\circ}(G)$ lie in these algebras.

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- [GR99] noted that in the case of association schemes, this allowed us to transform the underlying SDP into an LP by means of a common eigenbasis for all matrices of the algebra.
- [BGSV12] strengthened this method, showing how to explore symmetry and regularity in certain algebras in order to reduce the complexity of certain SDPs.

FCC

Theorem 1 (H. Assumpção, G. Coutinho)

If G is a k-regular graph and whose adjacency matrix A belongs to an association scheme, and if $\lambda_{min}(A)$ is its smallest eigenvalue, then

$$\eta^{\circ}(G) = \frac{2k}{k - \lambda_{min}(A)}.$$

In particular, we have

$$\frac{2k}{k - \lambda_{min}(A)} \leq fcc(G) \leq \frac{1}{\alpha_{GW}} \left(\frac{2k}{k - \lambda_{min}(A)} \right).$$

MAX 2-SAT

- We now consider two graphs G_1, G_2, with laplacian and signless laplacian matrices L, K, respectively. The program

$$\mathrm{qp}(G_1, G_2) := \max\left\{\left\langle \frac{L+K}{2}, xx^T \right\rangle : x \in \mathbb{R}^V, x_i^2 = 1\right\}$$

can be used to model MAX 2-SAT: given a boolean formula where each clause has precisely two literals, maximize the number of clauses that can be satisfied by an assignment.

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  can be used to model MAX 2-SAT: given a boolean formula where each clause has precisely two literals, maximize the number of clauses that can be satisfied by an assignment.
- Similarly to what was done with MAXCUT, we can consider

$$\gamma(G_1, G_2) := \max\left\{\left\langle \frac{L+K}{2}, M \right\rangle : M \succcurlyeq 0, \text{diag}(M) = \mathbb{1}\right\},$$

  and indeed the authors of [GW95] also show how to use this SDP to approximate MAX 2-SAT.

MAX 2-SAT

Theorem 2 (H. Assumpção, G. Coutinho)

If G_1, G_2 are graphs whose adjacency matrices A_1, A_2 belong to an association scheme with first eigenmatrix P, then

$$\gamma(G_1, G_2) = \frac{|V|}{2}\left((k_1 + k_2) + \max_{0 \leq l \leq d}\{P_{l2} - P_{l1}\}\right).$$

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This result combined with the approximation algorithm given in [GW95] provides a spectral bound for MAX 2-SAT in terms of the eigenvalues of the scheme.

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We can similarly obtain the dual parameter $\gamma^\circ(G_1, G_2)$ to $\gamma(G_1, G_2)$, and we can explicitly compute it in some cases.

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Theorem 3 (H. Assumpção, G. Coutinho)

Let G_1 be a distance-regular graph with diameter d and let G_2 be its distance-2 graph, with respective adjacency matrices A_1 and A_2. If P is the first eigenmatrix associated with the symmetric scheme generated by A_1, then

$$\gamma^\circ(G_1, G_2) = \begin{cases} \frac{k_1}{k_1 - P_{d1}}, & \text{if } k_2 P_{d1} + k_1 P_{d2} > 0, \\ \frac{k_1}{k_1 - P_{d1}} - \frac{(k_2 P_{d1} + k_1 P_{d2})}{2k_2(k_1 - P_{d1})}, & \text{otherwise.} \end{cases}$$

References I

- H. Assumpção and G. Coutinho, *Semidefinite programming bounds on fractional cut-cover and maximum 2-sat for highly regular graphs*, 2025.
- C. Bachoc, D. C. Gijswijt, A. Schrijver, and F. Vallentin, *Invariant semidefinite programs*, Handbook on Semidefinite, Conic and Polynomial Optimization (M. F. Anjos and J. B. Lasserre, eds.), Springer US, New York, NY, 2012, pp. 219–269.
- N. B. Proença, M. K. de Carli Silva, C. M. Sato, and L. Tunçel, *A primal-dual extension of the Goemans–Williamson algorithm for the weighted fractional cut-covering problem*, Mathematical Programming (2025), 1–56.

References II

- M. X. Goemans and F. Rendl, *Semidefinite programs and association schemes*, Computing **63** (1999), no. 4, 331–340.
- M. X. Goemans and D. P. Williamson, *Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming*, J. ACM **42** (1995), no. 6, 1115–1145.

Thank you!

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