Total Conformal Rigidity in Graphs

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Summary

1 Rigidity and Embeddings

2 Laplacian Cospectrality

3 SDPs

4 Laplacian Walks and Line Graphs

5 Walk-Regularity and Biregularity

Rigidity and Embeddings

Preliminaries

- $G = (V, E)$ be a simple, connected, undirected graph on n vertices with at least one edge, $A := A(G)$ is the adjacency matrix. For an edge $ab \in E$, let

$$z_{ab} := e_a - e_b,$$

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  and let $L_{ab} := z_{ab} z_{ab}^T$.
- The Laplacian is the linear operator $\mathcal{L} : \mathbb{R}^E \to \mathbb{S}^V$ such that

$$\mathcal{L}(w) := \sum_{ab \in E} w_{ab} L_{ab} = \sum_{ab \in E} w_{ab} z_{ab} z_{ab}^T, \qquad (1)$$

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- The simplex of valid edge weights is thus defined as

$$\Delta_E := \{w \in \mathbb{R}^E : w \geq 0,\ \mathbb{1}^T w = |E|\}.$$

Conformal Rigidity

Definition 1

We say that a graph G is *conformally rigid* if for any $w, w' \in \Delta_E$, we have

$$\lambda_2(w) \leq \lambda_2(\mathbb{1}) \leq \lambda_n(\mathbb{1}) \leq \lambda_n(w'),$$

that is, if $\mathbb{1}$ maximizes the second smallest eigenvalue of $\mathcal{L}(w)$, and minimizes its largest eigenvalue over all valid edge weights. If the former holds, we say the graph is *lower conformally rigid*, and if the latter holds, we say it is *upper conformally rigid*.

Some results

Some results from [ST25, GST25]:

- All 1-walk-regular and edge-transitive graphs are conformally rigid;
- G is conformally rigid iff the canonical embeddings onto $\mathcal{E}_{\lambda_2}, \mathcal{E}_{\lambda_n}$ are edge-isometric;
- Some sufficient conditions for Cayley graphs and vertex transitive graphs are also given.

k-Conformal rigidity

Definition 2

Let $S_k(w)$ and $s_k(w)$ denote the sum of the k largest and k smallest nontrivial eigenvalues of $\mathcal{L}(w)$, respectively, for some $k \in [n-1] := \{1, \ldots, n-1\}$. We say that G is *k-conformally rigid* if, for any $w, w' \in \Delta_E$,

$$s_k(w) \le s_k(\mathbb{1}) \le S_k(\mathbb{1}) \le S_k(w').$$

If $\mathbb{1}$ minimizes $S_k(w)$ over Δ_E, we say that G is *upper k-conformally rigid*, and if it maximizes $s_k(w)$, we say it is *lower k-conformally rigid*. If G is k-conformally rigid for all $k \in [n-1]$, we say that G is *totally conformally rigid*.

Embeddings

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Definition 3

Let $U \in \mathbb{R}^{n \times m}$ be a matrix whose columns form an orthonormal basis for $\mathcal{E}_\lambda$. The collection of vectors $\mathcal{U} = \{u_1, \ldots, u_n\} \subset \mathbb{R}^m$, where $u_i = U^T e_i$ is the i-th row of U, is called the *canonical spectral embedding* of G onto $\mathcal{E}_\lambda$.

Embeddings

Definition 4 (Edge-isometric embedding)

An embedding $\mathcal{U}$ is *edge-isometric* if there exists a constant $\gamma > 0$ such that $\|u_a - u_b\|^2 = \gamma$ for all $ab \in E$.

Embeddings

Definition 4 (Edge-isometric embedding)

An embedding $\mathcal{U}$ is *edge-isometric* if there exists a constant $\gamma > 0$ such that $\|u_a - u_b\|^2 = \gamma$ for all $ab \in E$.

The squared distance between adjacent vertices a and b is given by

$$\|u_a - u_b\|^2 = \|u_a\|^2 + \|u_b\|^2 - 2u_a^T u_b = X_{aa} + X_{bb} - 2X_{ab} = \mathcal{L}^*(X)_{ab},$$

hence the canonical embedding onto $\mathcal{E}_\lambda$ is edge-isometric if and only if $\mathcal{L}^*(X) = \gamma \mathbb{1}$ for some constant $\gamma > 0$.

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Definition 5

We say that a graph G is *edge-rigid* if all canonical embeddings onto the Laplacian eigenspaces are edge-isometric, that is, for every eigenprojector E_i of L corresponding to a nonzero eigenvalue there exists $\gamma_i > 0$ such that $\mathcal{L}^*(E_i) = \gamma_i \mathbb{1}$.

Laplacian Cospectrality

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- Two graphs G and H are called *cospectral* if their adjacency matrices share the same characteristic polynomial.
- We say that vertices u and v are cospectral if the characteristic polynomials of $A(G-u)$ and $A(G-v)$ are the same [GM80].
- The class of *walk-regular* graphs is precisely the class in which every pair of vertices is cospectral. Walk-regularity was then generalized by Dalfó, Fiol and Garriga in [DFG09].

Walk-regularity

Definition 6 (k-Walk-regularity)

We say that a graph G of diameter d is *k-walk-regular*, with $0 \leq k \leq d$, if the number of walks of length ℓ between any two vertices at distance i is a constant depending only on ℓ and i, for all integers $\ell \geq 0$ and $0 \leq i \leq k$.

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From this definition, it follows that 0-walk-regular graphs are precisely the class of walk-regular graphs, and d-walk-regular graphs are the class of distance-regular graphs.

Edge cospectrality

Given an edge $ab \in E$, the matrix $L - L_{ab}$ is the Laplacian of the graph $G - ab$ obtained by removing the edge ab while retaining the vertices a and b.

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Definition 7

Two edges $ab, cd \in E$ are called *Laplacian-cospectral* if the matrices

$$L - L_{ab} \quad \text{and} \quad L - L_{cd}$$

have the same characteristic polynomial.

Characterization

Theorem 8

All edges of G are pairwise Laplacian-cospectral if and only if G is edge-rigid.

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Corollary 10

If G is walk-regular, then it is edge-rigid if and only if it is 1-walk-regular.

Examples

Example 11

There are edge-rigid graphs that are not edge-transitive. Indeed, every strongly regular graph is 1-walk-regular, and by the result above every strongly regular graph is edge-rigid. Consequently, any strongly regular graph that is not edge-transitive gives an example. For instance, the Chang graphs [Cha59] on 28 vertices are strongly regular with parameters $(28, 12, 6, 4)$ and are not edge-transitive.

Examples

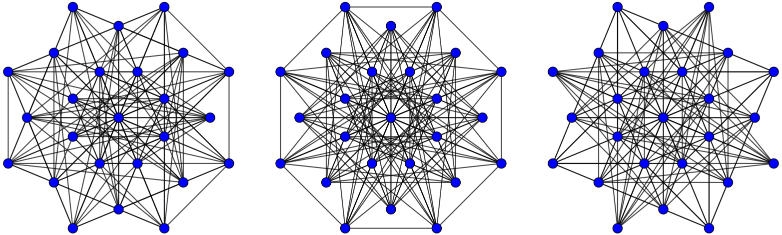

Figure: The three Chang graphs [Cha59], which are edge-rigid but not edge-transitive.

SDPs

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Define the following sets of positive semidefinite matrices:

$$\mathcal{X}_k := \{X \in \mathbb{S}^n : 0 \preceq X \preceq I,\ \mathrm{tr}(X) = k\},$$
$$\mathcal{Z}_k := \{Z \in \mathcal{X}_k : Z\mathbb{1}_V = 0\},$$

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$$s_k(w) = \min_{Z \in \mathcal{Z}_k} \mathrm{tr}(\mathcal{L}(w)Z). \tag{3}$$

Optimizing weights

Now consider the problem of minimizing $S_k(w)$ over all valid edge weights:

$$\min_{w \in \Delta_E} S_k(w) = \min_{w \in \Delta_E} \max_{X \in \mathcal{X}_k} \operatorname{tr}(\mathcal{L}(w) X).$$

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The dual of (2) is:

$$\min_{w \in \Delta_E} S_k(w) = \min\{ky + \operatorname{tr}(Y) : \quad yI + Y - \mathcal{L}(w) \succeq 0,\ Y \succeq 0,\ y \in \mathbb{R},\ w \in \Delta_E\}. \tag{4}$$

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The dual of the dual is:

$$\max\{|E|x : \mathcal{L}^*(X) \geq x\mathbb{1},\ X \in \mathcal{X}_k,\ x \in \mathbb{R}\}. \tag{5}$$

Necessity Lemma

Lemma 12

Suppose G is a graph with unweighted Laplacian $L = \sum_{i=1}^{r} \lambda_i E_i$, with $0 = \lambda_1 < \ldots < \lambda_r$. Fix $j \in [r-1]$, and let $k_j = \sum_{i=r-j+1}^{r} tr(E_i)$ and $X_j = \sum_{i=r-j+1}^{r} E_i$. Then the following hold:

1. *If G is upper k_j-conformally rigid, then $\mathcal{L}^*(X_j) = \beta \mathbb{1}$ for some $\beta > 0$.*
2. *If $j \in [r-2]$ and G is both upper k_j-conformally rigid and upper k_{j+1}-conformally rigid, then the canonical embedding onto the $(r-j)$-th eigenspace of L is edge-isometric. Moreover, G is also upper k-conformally rigid for all intermediate $k_j < k < k_{j+1}$.*

Analogous versions of these results hold for lower conformal rigidity.

Characterization

Theorem 13

A graph G is totally conformally rigid if and only if it is edge-rigid, that is, $\mathcal{L}^*(E_i)$ is a constant vector for every Laplacian eigenprojector E_i.

Consequences

Corollary 14

For a graph G, the following are equivalent:

(a) All edges of G are pairwise Laplacian-cospectral.
(b) G is edge-rigid.
(c) G is totally conformally rigid.
(d) For every Laplacian eigenprojector E_i, the vector $\mathcal{L}^*(E_i)$ is constant.

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(d) For every Laplacian eigenprojector E_i, the vector $\mathcal{L}^*(E_i)$ is constant.

Corollary 15

A graph G is upper k-conformally rigid for all $k \in [n-1]$ if and only if it is lower k-conformally rigid for all $k \in [n-1]$.

Laplacian Walks and Line Graphs

Characterization

Theorem 16

A graph G is edge-rigid if and only if for every integer $\ell \geq 0$, there exists a constant $C_\ell \in \mathbb{R}$ such that

$$\mathcal{L}^*(L^\ell)_{ab} = L^\ell_{aa} + L^\ell_{bb} - 2L^\ell_{ab} = C_\ell \quad \text{for every } ab \in E.$$

Characterization

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Corollary 17

If $G = (V, E)$ is a graph, then there is a polynomial-time algorithm for deciding if G is edge-rigid using $O(n^4)$ operations over the integers.

Line graphs

- Fix an arbitrary orientation of the edges of G.

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- Reversing edge orientations correspond to vertex switching, thus the diagonal entries of $(A_\sigma^\ell)_{ee}$ are invariant to orientation.

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- Reversing edge orientations correspond to vertex switching, thus the diagonal entries of $(A_\sigma^\ell)_{ee}$ are invariant to orientation.

Corollary 18

A graph G is edge-rigid if and only if, for any orientation of its edges, the corresponding signed line graph of G is walk-regular.

Walk-Regularity and Biregularity

Structural results

Lemma 19

If G is edge-rigid, then the sum of the degrees of the endpoints of any edge is a global constant across the graph. Consequently, G must be either regular or biregular bipartite.

Regular case

Theorem 20

If G is a regular edge-rigid graph, then G is walk-regular.

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Corollary 21

If G is a regular graph, then it is edge-rigid if and only if it is 1-walk-regular.

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If G is a regular graph, then it is edge-rigid if and only if it is 1-walk-regular.

A graph is equiarboreal if the number of spanning trees containing any given edge is independent of the choice of edge. Every 1-walk-regular graph is equiarboreal [**?**].

Corollary 22

Every regular edge-rigid graph is equiarboreal.

∎

Bipartite case

Definition 23 (Walk-biregularity)

Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. We say G is *walk-biregular* if, for every integer $\ell \geq 0$, there exist constants $\alpha_\ell^{(1)}$ and $\alpha_\ell^{(2)}$ such that the number of closed walks of length ℓ starting at any vertex $a \in V_1$ is exactly $\alpha_\ell^{(1)}$, and starting at any vertex $b \in V_2$ is exactly $\alpha_\ell^{(2)}$. Furthermore, we say G is 1-*walk-biregular* if it is walk-biregular and the number of walks of length ℓ between the endpoints of any edge $ab \in E$ is a global constant β_ℓ depending only on ℓ.

Bipartite case

Theorem 24

If $G = (V_1 \cup V_2, E)$ *is a biregular bipartite edge-rigid graph, then* G *is walk-biregular.*

Bipartite case

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Corollary 25

Let G be a walk-biregular graph. Then G is edge-rigid if and only if it is 1-walk-biregular.

Bipartite case

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If $G = (V_1 \cup V_2, E)$ is a biregular bipartite edge-rigid graph, then G is walk-biregular.

Corollary 25

Let G be a walk-biregular graph. Then G is edge-rigid if and only if it is 1-walk-biregular.

Corollary 26

A graph G is edge-rigid if and only if it is either 1-walk-regular or 1-walk-biregular.

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Questions?

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